Identification of Dynamical Systems
Master Course

Rik Pintelon

Vrije Universiteit Brussels, dept. ELEC

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### 1. Preliminary Example

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1. Preliminary Example

1.a Measurement of the value of a resistor

\[ R_0 = 1 \, \Omega \]

\[ i(k) = i_0 + n_i(k) \]

\[ u(k) = u_0 + n_u(k) \]

\[ n_i(k) \in N(0, \sigma_i^2) \]

\[ n_u(k) \in N(0, \sigma_u^2) \]

\[ R(k) = \frac{u(k)}{i(k)} \]
1. Preliminary Example

1.a Measurement of the value of a resistor (cont’d)

\[ \hat{R}_{SA}(N) = \frac{1}{N} \sum_{k=1}^{N} R(k) = \frac{1}{N} \sum_{k=1}^{N} \frac{u(k)}{i(k)} \]

Observation: the estimate seems not to converge as \( N \to \infty \)

Question: what is going wrong?
1. Preliminary Example

1.b Least squares estimator of the output error problem

\[ V_{LS}(R, z) = \sum_{k=1}^{N} (u(k) - Ri(k))^2 \]

\[ \hat{R}_{LS}(N) = \arg \min_{R} V_{LS}(R, z) = \frac{\sum_{k=1}^{N} u(k)i(k)}{\sum_{k=1}^{N} i^2(k)} \]

Observation: the estimate converges to a too small value as \( N \to \infty \)

Question: what is going wrong?
1. Preliminary Example

1.c Maximum likelihood estimator of the errors-in-variables problem

Call \( z \) the vector of the measurements
\[
z = [u(1), u(2), \ldots, u(N), i(1), i(2), \ldots, i(N)]^T
\]

Gaussian iid random variables \( \Rightarrow \) probability density function of the measurements \( z \)
\[
f_z(z|R, u_p, i_p) = \prod_{k=1}^{N} f_{u(k)}(u(k)|R, u_p, i_p)f_{i(k)}(i(k)|R, u_p, i_p)
\]
\[
= \frac{1}{(2\pi \sigma_u \sigma_i)^N} \exp\left(-\frac{1}{2} \sum_{k=1}^{N} \frac{(u(k) - u_p)^2}{\sigma_u^2} + \frac{(i(k) - i_p)^2}{\sigma_i^2}\right)
\]

Constrained negative log-likelihood function
\[
-\log f_z(z|R, u_p, i_p) = \text{constant} + \frac{1}{2} \sum_{k=1}^{N} \frac{(u(k) - u_p)^2}{\sigma_u^2} + \frac{(i(k) - i_p)^2}{\sigma_i^2} \quad \text{s.t. } u_p = Ri_p
\]

Maximum likelihood cost function
\[
V_{ML}(R, u_p, i_p, z) = \frac{1}{2} \sum_{k=1}^{N} \frac{(u(k) - u_p)^2}{\sigma_u^2} + \frac{(i(k) - i_p)^2}{\sigma_i^2} + \lambda(u_p - Ri_p)
\]

Eliminate \( u_p, i_p \) and \( \lambda \) \( \Rightarrow \hat{R}_{ML}(N) \)
1. Preliminary Example

1.c Maximum likelihood estimator of the errors-in-variables problem (cont’d)

\[ \hat{R}_{ML}(N) = \frac{1}{N} \sum_{k=1}^{N} u(k) \]

Observation: estimate converges to the true value as \( N \to \infty \)
1. Preliminary Example

1.c Maximum likelihood estimator of the errors-in-variables problem (cont’d)

\[ \hat{R}_{\text{ML}}^{[r]}(N) = \frac{1}{N} \sum_{k=1}^{N} u^{[r]}(k) \]
\[ = \frac{1}{N} \sum_{k=1}^{N} i^{[r]}(k) \]

\( r = 1, 2, \ldots \)

\[ R_{\text{ML}}^{[r]}(N) \]
\[ \lim_{N \to \infty} R_{\text{ML}}^{[r]}(N) = R_{\text{ML}}^{[r]} \]

Observation: estimate is different for each realisation \( \Rightarrow \) stochastic process

Questions:
- point wise convergence \( \lim_{N \to \infty} \hat{R}_{\text{ML}}^{[r]}(N) = R_{\text{ML}}^{[r]} \) ?
- \( R_{\text{ML}}^{[r]} = R_{\text{ML}} \) “for almost all realisations”?
- \( R_{\text{ML}} = R_0 ? \)
- minimal uncertainty?
1. Preliminary Example

1.d Three modes of stochastic convergence

1. convergence with probability one

\[ \text{Prob}(\lim_{N \to \infty} \hat{R}(N) = R_0) = 1 \iff \lim_{N \to \infty} \hat{R}(N) = R_0 \text{ w.p. 1} \iff \text{a.s.} \lim_{N \to \infty} \hat{R}(N) = R_0 \]

2. convergence in probability

\[ \forall \varepsilon > 0: \lim_{N \to \infty} \text{Prob}(|\hat{R}(N) - R_0| \leq \varepsilon) = 1 \iff \lim_{N \to \infty} \hat{R}(N) = R_0 \text{ in prob.} \iff \text{plim} \hat{R}(N) = R_0 \]

3. convergence in mean square sense

\[ \lim_{N \to \infty} \mathbb{E} \{ |\hat{R}(N) - R_0|^2 \} = 0 \iff \lim_{N \to \infty} \hat{R}(N) = R_0 \text{ in mean square} \iff \text{l.i.m.} \hat{R}(N) = R_0 \]

Notes

- stochastic convergence does not guarantee convergence for all realisations
- for infinitely many realisations the limit might not exist
1. Preliminary Example

1.d Three modes of stochastic convergence (cont’d)

Properties

- a.s.lim and plim can be interchanged with a continuous function
- l.i.m. cannot be interchanged with a continuous function
- l.i.m. of a linear combination = linear combination of the l.i.m.

Interrelations

- a.s.lim and l.i.m. imply plim; the converse is not true
- there is no implication between a.s.lim and l.i.m.
1. Preliminary Example

1.e Properties estimators

- stochastic convergence
- asymptotic variance
- asymptotic distribution
- asymptotic efficiency
- robustness

**Question:** how to predict these properties?

\[
\hat{R}_{SA}(N) = \frac{1}{N} \sum_{k=1}^{N} \frac{u(k)}{i(k)}
\]

\[
\hat{R}_{LS}(N) = \frac{\sum_{k=1}^{N} u(k)i(k)}{\sum_{k=1}^{N} i^2(k)}
\]

\[
\hat{R}_{ML}(N) = \frac{\sum_{k=1}^{N} u(k)}{\frac{1}{N} \sum_{k=1}^{N} i(k)}
\]
1. Preliminary Example

1.6 Stochastic convergence estimators

Basic tool: the *law of large numbers*

\[ S(N) = \frac{1}{N} \sum_{k=1}^{N} x(k) \] then \( S(N) \to \mathbb{E} \{ S(N) \} = \frac{1}{N} \sum_{k=1}^{N} \mathbb{E} \{ x(k) \} \) as \( N \to \infty \)

- Simple approach

\[ \hat{R}_{SA}(N) = \frac{1}{N} \sum_{k=1}^{N} \frac{u(k)}{i(k)} \to \frac{1}{N} \sum_{k=1}^{N} \mathbb{E} \{ u(k) \} \mathbb{E} \left\{ \frac{1}{i(k)} \right\} = \infty \text{ for } i(k) \in N(i_0, \sigma_i^2) \]

\( \Rightarrow \) no convergence

- Least squares

\[ \hat{R}_{LS}(N) = \frac{\frac{1}{N} \sum_{k=1}^{N} u(k) i(k)}{\frac{1}{N} \sum_{k=1}^{N} i^2(k)} \to \frac{\frac{1}{N} \sum_{k=1}^{N} \mathbb{E} \{ u(k) \} \mathbb{E} \{ i(k) \}}{\frac{1}{N} \sum_{k=1}^{N} \mathbb{E} \{ i^2(k) \}} = \frac{u_0i_0}{i_0^2 + \sigma_i^2} = \frac{R_0}{1 + \sigma_i^2/i_0^2} \]

\( \Rightarrow \) convergence to the wrong value = inconsistent

- Maximum likelihood

\[ \hat{R}_{ML}(N) = \frac{\frac{1}{N} \sum_{k=1}^{N} u(k)}{\frac{1}{N} \sum_{k=1}^{N} i(k)} \to \frac{\frac{1}{N} \sum_{k=1}^{N} \mathbb{E} \{ u(k) \}}{\frac{1}{N} \sum_{k=1}^{N} \mathbb{E} \{ i(k) \}} = \frac{u_0}{i_0} = R_0 \]

\( \Rightarrow \) convergence to the true value = consistent
1. Preliminary Example

1.g Asymptotic variance estimators

Basic tool: linearisation estimate around the limit value

- Least squares

\[ 
\hat{R}_{LS}(N) = \frac{1}{N} \sum_{k=1}^{N} \frac{u(k)i(k)}{i_0^2 + w[k]} = \frac{u_0i_0 + w[1]}{i_0^2 + w[2]} \approx \frac{R_0}{1 + \sigma_i^2/i_0^2} \left( 1 + \frac{w[1]}{u_0i_0} - \frac{w[2] - \sigma_i^2}{i_0^2 + \sigma_i^2} \right) 
\]

\[ 
w[1] = \frac{1}{N} \sum_{k=1}^{N} (u_0n_i(k) + i_0n_u(k) + n_i(k)n_u(k)) \quad \text{with} \quad \mathbb{E}\{w[1]\} = 0 
\]

\[ 
w[2] = \frac{1}{N} \sum_{k=1}^{N} (2i_0n_i(k) + n_i^2(k)) \quad \text{with} \quad \mathbb{E}\{w[2]\} = \sigma_i^2 
\]

\[ 
\Rightarrow \text{the asymptotic } (N \to \infty) \text{ variance} 
\]

\[ 
"\text{var}(\sqrt{N}\hat{R}_{LS}(N))" \rightarrow \frac{R_0^2}{(1 + \sigma_i^2/i_0^2)^2} \text{var}(\sqrt{N}\left(\frac{w[1]}{u_0i_0} - \frac{w[2] - \sigma_i^2}{i_0^2 + \sigma_i^2}\right)) 
\]

Notes:

- valid for any voltage and current signal-to-noise ratio
- asymptotic variance is an \( O(N^{-1}) \)
1. Preliminary Example

1.g Asymptotic variance estimators (cont’d)

• Maximum likelihood

\[ \hat{R}_{ML}(N) = \frac{1}{N} \sum_{k=1}^{N} u(k) - \frac{1}{N} \sum_{k=1}^{N} i(k) = \frac{u_0 + w_{[1]}}{i_0 + w_{[2]}} \approx R_0 \left( 1 + \frac{w_{[1]}}{u_0} - \frac{w_{[2]}}{i_0} \right) \]

\[ w_{[1]} = \frac{1}{N} \sum_{k=1}^{N} n_u(k) \text{ with } \mathbb{E}\{w_{[1]}\} = 0 \]

\[ w_{[2]} = \frac{1}{N} \sum_{k=1}^{N} n_i(k) \text{ with } \mathbb{E}\{w_{[2]}\} = 0 \]

\[ \Rightarrow \text{the asymptotic } (N \to \infty) \text{ variance} \]

\[ "\text{var}(\sqrt{N}\hat{R}_{ML}(N))" \to R_0^2 \text{var}(\sqrt{N}\left( \frac{w_{[1]}}{u_0} - \frac{w_{[2]}}{i_0} \right)) = R_0^2 \left( \frac{\sigma_i^2}{i_0^2} + \frac{\sigma_u^2}{u_0^2} \right) \]

Notes:

- valid for any voltage and current signal-to-noise ratio
- asymptotic variance is an \( O(N^{-1}) \)
- expected value \( \hat{R}_{ML}(N) \) does not exist
- variance \( \hat{R}_{ML}(N) \) does not exist
1. Preliminary Example

1.g Asymptotic variance estimators (cont’d)

\[
\frac{\sigma_u}{u_0} = \frac{\sigma_i}{i_0}
\]

\[
\text{MSE}(\hat{R}) = \mathbb{E} \{ (\hat{R} - R_0)^2 \} = \text{var}(\hat{R}) + (\mathbb{E} \{ \hat{R} \} - R_0)^2
\]
1. Preliminary Example

1.h Asymptotic normality estimators

Basic tool: linearisation estimate around the limit value + the central limit theorem

\[ S(N) = \frac{1}{N} \sum_{k=1}^{N} x(k) \text{ then } \frac{S(N) - \mathbb{E}\{S(N)\}}{\sqrt{\text{var}(S(N))}} \in \text{AsN}(0, 1) \text{ as } N \to \infty \]

- Least squares and maximum likelihood estimators

\[ \hat{R}_{LS}(N) \approx \frac{R_0}{1 + \sigma_i^2 / \sigma_i^2} \left( 1 + \frac{w[1]}{u_0i_0} - \frac{w[2] - \sigma_i^2}{i_0^2 + \sigma_i^2} \right) \]

\[ \hat{R}_{ML}(N) \approx R_0 \left( 1 + \frac{w[1]}{u_0} - \frac{w[2]}{i_0} \right) \]

- Application CLT to \( w[1], w[2] \Rightarrow w[1], w[2] \) are asymptotically normally distributed \( \Rightarrow \)

\[ \hat{R}_{LS}(N), \hat{R}_{ML}(N) \text{ are asymptotically normally distributed} \]
1. Preliminary Example

1.h Asymptotic normality estimators (cont’d)

\[ N = 10 \quad \quad \quad N = 100 \quad \quad \quad N = 1000 \]

\[ R \]

\[ \hat{R}_{LS} \quad \hat{R}_{ML} \]
1. Preliminary Example

1.i Asymptotic efficiency estimators

Basic tool: comparison asymptotic uncertainty with the Cramér-Rao lower bound

Gaussian iid random variables \( \Rightarrow \) probability density function of the measurements \( z \)

\[
f_z(z \mid \theta) = \prod_{k=1}^{N} f_{u(k)}(u(k) \mid \theta) f_{i(k)}(i(k) \mid \theta)
\]

\[
= \frac{1}{(2\pi \sigma_u \sigma_i)^N} \exp \left( -\frac{1}{2} \sum_{k=1}^{N} \frac{(u(k) - R_i p)^2}{\sigma_u^2} + \frac{(i(k) - i_p)^2}{\sigma_i^2} \right)
\]

with \( \theta = [i_p, R]^T \)

The Cramér-Rao lower bound

\[
\text{Cov}(\hat{\theta}) \geq Fi^{-1}(\theta_0) \text{ with } Fi(\theta_0) = -\mathcal{E} \left\{ \frac{\partial^2 \log f_z(z \mid \theta)}{\partial \theta_0^2} \right\}
\]

the corresponding Fisher information matrix

\[
Fi^{-1}(\theta_0) = \frac{1}{N} \begin{bmatrix}
\sigma_i^2 & -u_0 \frac{\sigma_i^2}{i_0^2} \\
-u_0 \frac{\sigma_i^2}{i_0^2} & R_0^2 \left( \frac{\sigma_i^2}{i_0^2} + \frac{\sigma_u^2}{u_0^2} \right)
\end{bmatrix}
\Rightarrow Fi^{-1}(R_0) = \frac{R_0^2}{N} \left( \frac{\sigma_i^2}{i_0^2} + \frac{\sigma_u^2}{u_0^2} \right)
1. Preliminary Example

1.i Asymptotic efficiency estimators (cont’d)

Conclusion: ML-estimate is asymptotically efficient

\[
\text{std}(\hat{R}_{ML}) + \sqrt{CR(R_0)} = Fi^{-1/2}(R_0)
\]

\[
\frac{\sigma_u}{u_0} = \frac{\sigma_i}{i_0}
\]
1. Preliminary Example

1.j Robustness estimators

Basic question: are the (asymptotic) properties of the estimator sensitive to the (noise) assumptions made to construct the estimator?

• Maximum likelihood estimator
  - basic assumption: zero mean Gaussian iid noise on current and voltage measurements
  - relax assumption: zero mean non-Gaussian mixing noise on current and voltage measurements

⇒ consequences on the properties:
  - consistency, asymptotic normality, convergence rate still valid
  - asymptotic efficiency is lost
  - expression asymptotic variance still valid for independently distributed noise

• Least squares estimator
  - basic assumption: zero mean iid noise on voltage measurements, and current known exactly
  - relax assumption:
    (i) zero mean non-Gaussian mixing noise on voltage measurements, and current known exactly ⇒ still consistent
    (ii) also noise on current ⇒ inconsistent
1. Preliminary Example

1.j Robustness estimators (cont’d)

\[u(k) = u_0 + n_u(k)\]
\[i(k) = i_0 + n_i(k)\]

\[u_0 = 1 \text{ V}\]
\[i_0 = 1 \text{ A}\]

\[n_u(k) \in [-0.5 \text{ V}, 0.5 \text{ V}] \text{ uniformly}\]
\[n_i(k) \in [-0.5 \text{ A}, 0.5 \text{ A}] \text{ uniformly}\]

Note:

\[\mathbb{E}\{\hat{R}_{SA}(N)\} = R_0 \frac{i_0}{2\sqrt{3}\sigma_i} \log \left( \frac{1 + \sqrt{3}\sigma_i / i_0}{1 - \sqrt{3}\sigma_i / i_0} \right) = 1.1 < \infty\]

\[\hat{R}_{LS}(N) \rightarrow \frac{R_0}{1 + \sigma_i^2 / i_0^2} = 0.92\]
1. Preliminary Example

1.k Summary properties

• Goal asymptotic analysis
  - what happens with the estimate if more data are gathered?
  - hypothesis: the asymptotic behaviour reflects the finite sample behaviour
  - true finite sample behaviour is in general very difficult to establish
    (exception: linear least squares)

• Consistency
  - convergence in *stochastic* sense to the true value
  - does not exclude divergence for some realisations
  - guarantees with high probability that the estimate is close to the true value
  - consistency does not imply asymptotic unbiasedness
  - proof: law of large numbers

• Asymptotic unbiasedness
  - asymptotically the expected value equals the true value
  - does not guarantee that the estimate is close to the true value
  - in general very difficult to verify (exception: linear least squares)

• Asymptotic normality
  - allows to construct uncertainty bounds with a given confidence level
  - proof: linearisation around limit value + central limit theorem
1. Preliminary Example

1.k Summary properties (cont’d)

• Asymptotic variance
  - measure of the convergence rate
  - construction of uncertainty bounds
  - proof: linearisation around limit value

• Asymptotic efficiency
  - minimal uncertainty within the class of asymptotically unbiased estimators
  - in practice applied to the class of consistent estimators
  - Cramér-Rao lower bound does not always exist (e.g. uniformly distributed noise), or may be too conservative (minimal uncertainty may be higher than the CR-bound)
  - proof: comparison asymptotic variance with Cramér-Rao lower bound

• Robustness
  - sensitivity (asymptotic) properties to the noise assumptions made to construct the estimator
  - proof: validity law of large numbers and central limit theorem under relaxed noise assumptions
1. Preliminary Example

1.1 Extension to arbitrary currents

Zero mean Gaussian iid random variables $\Rightarrow$ maximum likelihood cost function

$$V_{ML}(R, u_p, i_p, z) = \frac{1}{2} \sum_{k=1}^{N} \left( \frac{(u(k) - u_p(k))^2}{\sigma_u^2(k)} + \frac{(i(k) - i_p(k))^2}{\sigma_i^2(k)} + \sum_{k=1}^{N} \lambda_k (u_p(k) - Ri_p(k)) \right)$$

Elimination $u_p(k), i_p(k), \lambda_k, k = 1, 2, ..., N$ gives

$$V_{ML}(R, z) = \frac{1}{2} \sum_{k=1}^{N} \left( \frac{(u(k) - Ri(k))^2}{\sigma_u^2(k) + R^2 \sigma_i^2(k)} \right)$$

which is a weighted nonlinear least squares problem

- starting values via linear least squares
- iterative Newton-Gauss minimization procedure
1. Preliminary Example

1.1 Extension to arbitrary currents (cont’d)

- Consistency

Difficulty: no explicit expression for the minimizer $\hat{R}$ of $V_{ML}(R, z)$ exists

Strong law of large numbers applied to $V_N(R, z) = V_{ML}(R, z)/N$

$$V_N(R, z) \rightarrow V_N(R) = \mathbb{E}\{V_N(R, z)\} = \frac{1}{2N} \sum_{k=1}^{N} \frac{\mathbb{E}\{(u_0(k) - Ri_0(k))^2\}}{\sigma_u^2(k) + R^2 \sigma_i^2(k)} + \frac{1}{2}$$

Expected value cost function $V_N(R)$ is minimal in $R = R_0 \Rightarrow$ consistent
1. Preliminary Example

1.1 Extension to arbitrary currents (cont’d)

- Asymptotic variance and asymptotic normality

Mean value theorem on derivative cost function

\[ V_N'(\hat{R}, z) = V_N'(R_0, z) + V_N''(R_1, z)(\hat{R} - R_0) \] with \( R_1 = t\hat{R} + (1 - t)R_0, \ t \in [0, 1] \)

Use

\[ V_N'(\hat{R}, z) = 0 \] (by definition of \( \hat{R} \))

\( R_1 \to R_0 \) as \( N \to \infty \) (consistency)

\[ V_N''(R_1, z) \to V_N''(R_0) \] as \( N \to \infty \) (law of large numbers)

then

\[ \hat{R} - R_0 \to \delta_R = -V_N''^{-1}(R_0)V_N'(R_0, z) \text{ with } \mathbb{E}\{\delta_R\} = 0 \]

Asymptotic variance

\[ \text{var}(\hat{R}) \to \text{var}(\delta_R) = V_N''^{-2}(R_0)\mathbb{E}\{(V_N'(R_0, z))^2\} \]

Asymptotic normality

\[ \sqrt{N}(\hat{R} - R_0) \in \text{AsN}(0, \text{var}(\sqrt{N}\delta_R)) \] (central limit theorem on \( V_N'(R_0, z) \))
1. Preliminary Example

1.1 Extension to arbitrary currents (cont’d)

- Asymptotic efficiency

  Gaussian iid random variables ⇒ probability density function of the measurements $z$

  $$f_z(z|\theta) = \prod_{k=1}^{N} f_{u(k)}(u(k)|\theta) f_{i(k)}(i(k)|\theta)$$

  $$= \frac{1}{(2\pi)^N \prod_{k=1}^{N} \sigma_u(k) \sigma_i(k)} \exp\left(-\frac{1}{2} \sum_{k=1}^{N} \frac{(u(k) - R_i(k))^2}{\sigma_u^2(k)} + \frac{(i(k) - i_p(k))^2}{\sigma_i^2(k)}\right)$$

  with $\theta = [i_p(1), i_p(2), \ldots, i_p(N), R]^T$

  The Fisher information matrix equals

  $$Fi(\theta_0) = -\mathbb{E}\left\{\frac{\partial^2 \log f_z(z|\theta)}{\partial \theta_0^2}\right\} \Rightarrow Fi^{-1}(R_0) = \frac{1}{N} V_N''^{-1}(R_0)$$

  Conclusion

  - the ML estimate is in general inefficient
  - asymptotically efficient if current is known exactly

  $$\mathbb{E}\{(V_N'(R_0, z))^2\} = V_N''(R_0)/N \Rightarrow \text{var}(\delta_R) = Fi^{-1}(R_0)$$
1. Preliminary Example

1.1 Extension to arbitrary currents (cont’d)

- Notes
  - inefficiency ML estimate is not in contradiction with general ML properties because an \textit{infinite} number of parameters are estimated
  - uncertainty on \( i_p(1), i_p(2), \ldots, i_p(N) \) does not decrease as \( N \to \infty \)
  - in practice a good approximation of the variance of \( \hat{R} \) is given by

\[
\frac{1}{N} V_N^{n-1}(R_0) = \left[ \left( \frac{\partial \varepsilon(R, z_0)}{\partial R_0} \right)^T \left( \frac{\partial \varepsilon(R, z_0)}{\partial R_0} \right) \right]^{-1} \approx \left[ \left( \frac{\partial \varepsilon(R, z)}{\partial \hat{R}} \right)^T \left( \frac{\partial \varepsilon(R, z)}{\partial \hat{R}} \right) \right]^{-1}
\]

where

\[
\varepsilon(R, z) = \begin{bmatrix}
\varepsilon_{[1]}(R, z) \\
\varepsilon_{[2]}(R, z) \\
\vdots \\
\varepsilon_{[N]}(R, z)
\end{bmatrix}
\]

with \( \varepsilon_{[k]}(R, z) = \frac{u(k) - Ri(k)}{\sqrt{\sigma_i^2(k) + R^2 \sigma_e^2(k)}} \)

- the variance of the estimate follows as a by product of the Newton-Gauss minimization procedure
1. Preliminary Example

1.m References
2. Nonparametric models of LTI systems

2.a Problem statement
2.b Periodic excitations – errors-in-variables problem
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2.d Arbitrary excitations – output error problem
2.e Simulation example: FRF and variance estimate using arbitrary input
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2.i Simulation example: FRF estimate in the presence of nonlinear distortions
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2.k Nonlinear systems operating in feedback
2.l Measurement example: the open loop gain of an operational amplifier
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2.u MIMO experiment: identification d-axis synchronous machine
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2. Nonparametric models of LTI systems

2.a Problem statement

Noisy observations

\[ Y(k) = Y_0(k) + N_Y(k) \]
\[ U(k) = U_0(k) + N_U(k) \]

- periodic signals

\[ \begin{cases} Y_0(k) = G_0(j\omega_k)U_0(k) \\ U_0(k) = U_g(k) \end{cases} \quad \text{and} \quad \begin{cases} N_Y(k) = G_0(j\omega_k)N_g(k) + N_p(k) + M_Y(k) \\ N_U(k) = N_g(k) + M_U(k) \end{cases} \]

- arbitrary signals

\[ \begin{cases} Y_0(k) = G_0(j\omega_k)U_0(k) + T_G(j\omega_k) \\ U_0(k) = U_1(k) \end{cases} \quad \text{and} \quad \begin{cases} N_Y(k) = N_p(k) + M_Y(k) \\ N_U(k) = M_U(k) \end{cases} \]
2. Nonparametric models of LTI systems

2.a Problem statement (cont’d)

- periodic signals

\[
\begin{align*}
Y_0(k) &= G_0(j\omega_k)U_0(k) \\
U_0(k) &= \frac{1}{1 + G_0(j\omega_k)}R(k)
\end{align*}
\]

and

\[
\begin{align*}
N_Y(k) &= \frac{G_0(j\omega_k)N_g(k) + N_p(k)}{1 + G_0(j\omega_k)} + M_Y(k) \\
N_U(k) &= \frac{N_g(k) - N_p(k)}{1 + G_0(j\omega_k)} + M_U(k)
\end{align*}
\]

- arbitrary signals

\[
\begin{align*}
Y_0(k) &= G_0(j\omega_k)U_0(k) + T_G(j\omega_k) \\
U_0(k) &= U_1(k)
\end{align*}
\]

and

\[
\begin{align*}
N_Y(k) &= N_p(k) + M_Y(k) \\
N_U(k) &= M_U(k)
\end{align*}
\]

Ng(k) generator noise

Np(k) process noise

MU(k) input measurement noise

MY(k) output measurement noise
2. Nonparametric models of LTI systems

2.a Problem statement (cont’d)

\[
\begin{align*}
N_g(k) & \quad \text{generator noise} \\
N_p(k) & \quad \text{process noise} \\
M_U(k) & \quad \text{input measurement noise} \\
M_Y(k) & \quad \text{output measurement noise}
\end{align*}
\]

Conclusion

periodic excitation
- true input independent of the noise sources
- part plant input is discarded

arbitrary excitation
- true input depends on the process noise
Nonparametric models of LTI systems

2.a Problem statement (cont’d)

• Nonparametric representation

  Frequency response function
  \[ G_0(j\omega_k), \ k = 1, 2, \ldots \]

  (Co-)variances circular complex distributed noise
  \[
  \sigma_Y^2(k) = \text{var}(N_Y(k)) = \mathbb{E}\{|N_Y(k)|^2\}
  \]
  \[
  \sigma_U^2(k) = \text{var}(N_U(k)) = \mathbb{E}\{|N_U(k)|^2\}, \ k = 1, 2, \ldots
  \]
  \[
  \sigma_{YU}(k) = \text{covar}(N_Y(k), N_U(k)) = \mathbb{E}\{N_Y(k)\overline{N_U(k)}\}
  \]

  and
  \[
  \mathbb{E}\{N_Y^2(k)\} = 0
  \]
  \[
  \mathbb{E}\{N_U^2(k)\} = 0
  \]
  \[
  \mathbb{E}\{N_Y(k)N_U(k)\} = 0
  \]
2. Nonparametric models of LTI systems

2.b Periodic excitations – errors-in-variables problem

• Noisy input-output measurements

• Observe $M$ consecutive periods of the steady state response

$$m = 1 \quad 2 \quad \ldots \quad M$$

$N$ data points

$N_T = M \times N$ data points

Sample means

$$\hat{Y}(k) = \frac{1}{M} \sum_{m=1}^{M} Y^{[m]}(k), \quad \hat{U}(k) = \frac{1}{M} \sum_{m=1}^{M} U^{[m]}(k)$$

Sample (co-)variances

$$\hat{\sigma}_Y^2(k) = \frac{1}{M(M-1)} \sum_{m=1}^{M} \left| Y^{[m]}(k) - \hat{Y}(k) \right|^2, \quad \hat{\sigma}_U^2(k) = \frac{1}{M(M-1)} \sum_{m=1}^{M} \left| U^{[m]}(k) - \hat{U}(k) \right|^2,$$

$$\hat{\sigma}_{YU}^2(k) = \frac{1}{M(M-1)} \sum_{m=1}^{M} \left( Y^{[m]}(k) - \hat{Y}(k) \right) \left( U^{[m]}(k) - \hat{U}(k) \right)$$
2. Nonparametric models of LTI systems

2.b Periodic excitations – errors-in-variables problem (cont’d)

• Maximum likelihood estimate frequency response function

\[
Z[m] = [Y[m](k), U[m](k)]^T \text{ is independent (over } m\text{), circular complex normally distributed } \Rightarrow \\
\text{Gaussian probability density function of the measurements } Z
\]

\[
f_Z(Z|\theta) = \prod_{m=1}^{M} f_{Z[m]}(Z[m]|\theta)
\]

\[
= \frac{1}{\pi^{2M} \prod_{m=1}^{M} \text{det}(C_Z)} \exp(-\sum_{m=1}^{M} (Z[m] - Z_p)^H C_Z^{-1}(Z[m] - Z_p))
\]

with

\[
\theta = [Z_p^T, G(j \omega_k)]^T, Z_p = \begin{bmatrix} Y_p(k) \\ U_p(k) \end{bmatrix} \text{ and } C_Z = \text{Cov}(Z[m]) = \begin{bmatrix} \sigma_{\hat{Y}}^2(k) & \sigma_{\hat{Y}U}^2(k) \\ \sigma_{\hat{Y}U}^2(k) & \sigma_{\hat{U}}^2(k) \end{bmatrix}
\]

Gaussian ML cost function

\[
V_{\text{ML}}(G(j \omega_k), Z_p, Z) = \sum_{m=1}^{M} (Z[m] - Z_p)^H C_Z^{-1}(Z[m] - Z_p) + \lambda [1, -G(j \omega_k)] Z_p
\]

Eliminate \( Z_p \), \( \lambda \)

\[
\hat{G}_{\text{ML}}(j \omega_k) = \frac{\hat{Y}(k)}{\hat{U}(k)} = \frac{M^{-1} \sum_{m=1}^{M} Y[m](k)}{M^{-1} \sum_{m=1}^{M} U[m](k)}
\]
2. Nonparametric models of LTI systems

2.b Periodic excitations – errors-in-variables problem (cont’d)

- Maximum likelihood estimate frequency response function (cont’d)

Properties ML estimate FRF

\[
\hat{G}_{ML}(j\omega_k) = \frac{\hat{Y}(k)}{\hat{U}(k)} = \frac{1}{M} \sum_{m=1}^{M} \frac{Y[m](k)}{U[m](k)}
\]

- consistent \((M \to \infty)\)
- asymptotically circular complex normally distributed
- asymptotically efficient
- robust w.r.t. independence and normality assumptions
- expected value exists \(\Rightarrow\) bias expression
- variance does not exist

Cramér-Rao lower bound

\[
\text{var}(\hat{G}(j\omega_k)) \geq \left|G_0(j\omega_k)\right|^2 \left(\frac{\sigma_Y^2(k)}{|Y_0(k)|^2} + \frac{\sigma_U^2(k)}{|U_0(k)|^2} - 2\text{Re}\left(\frac{\sigma_{YU}^2(k)}{Y_0(k)\overline{U_0(k)}}\right)\right) \text{ where } \sigma_X^2 = O\left(\frac{1}{M}\right)
\]
2. Nonparametric models of LTI systems

2.b Periodic excitations – errors-in-variables problem (cont’d)

• Maximum likelihood estimate frequency response function (cont’d)

Relative bias on FRF estimate

\[ b(k) = \frac{\mathbb{E}\{\hat{G}(j\omega_k)\}}{G_0(j\omega_k)} - 1 = -\exp\left(-\frac{|U_0(k)|^2}{\sigma^2_U(k)}\right) \left(1 - \frac{\sigma^2_{\hat{U}}(k)}{\sigma_U(k)\sigma_Y(k)|Y_0(k)|/\sigma_Y(k)}\right) \]

where

\[ |b(k)| < 1 \times 10^{-4} \text{ if } \min\left(\frac{|U_0(k)|}{\sigma_U(k)}, \frac{|Y_0(k)|}{\sigma_Y(k)}\right) > 10 \text{ dB} \]

Asymptotic \((M \to \infty)\) variance

\(\hat{G}_{ML}(j\omega_k) \approx G_0(j\omega_k) + \delta_G(k)\)

with

\[ \delta_G(k) = G_0(j\omega_k)\left(\frac{1}{M} \sum_{m=1}^{M} N_\mathcal{U}^m(k) - \frac{1}{M} \sum_{m=1}^{M} N_\mathcal{Y}^m(k)\right) \]

\[ \text{var}(\delta_G(k)) = |G_0(j\omega_k)|^2 \left\{ \frac{\sigma^2_Y(k)}{|Y_0(k)|^2} + \frac{\sigma^2_U(k)}{|U_0(k)|^2} - 2\text{Re}\left(\frac{\sigma_{\hat{U}\hat{Y}}(k)}{Y_0(k)U_0(k)}\right) \right\} \]
2. Nonparametric models of LTI systems

2. Nonparametric models of LTI systems

2.b Periodic excitations – errors-in-variables problem (cont’d)

• Maximum likelihood estimate frequency response function (cont’d)

Uncertainty bounds

\[ \hat{G}_{ML}(j\omega_k) - G_0(j\omega_k) \in A_{sN_c}(0, \hat{\sigma}_G^2(k)) \]

with

\[ \hat{\sigma}_G^2(k) = \left| \hat{G}_{ML}(j\omega_k) \right|^2 \left( \frac{\hat{\sigma}_{\hat{Y}}^2(k)}{\left| \hat{Y}(k) \right|^2} + \frac{\hat{\sigma}_{\hat{U}}^2(k)}{\left| \hat{U}(k) \right|^2} - 2\text{Re}\left( \frac{\hat{\sigma}_{\hat{Y}\hat{U}}^2(k)}{\hat{Y}(k)\hat{U}(k)} \right) \right) \]

⇒ \( p \) % confidence region = circle with radius

\[ \sqrt{-\log(1-p)} \hat{\sigma}_G(k) \]

Notes:

- uncertainty bound accurate if \( |U_0(k)|/\sigma_{\hat{U}}(k) > 20 \text{ dB} \)

- \( p = 0.95 \) ⇒ radius \( \approx \sqrt{3} \hat{\sigma}_G(k) \)

- decrease uncertainty FRF estimate: \( M \) large for a given \( N_T = M \times N \)

- increase frequency resolution \( f_s/N: M \) small for a given \( N_T = M \times N \)
2. Nonparametric models of LTI systems

2.c Measurement example: octave bandpass filter

- multisine excitation $k f_0, k = 1, 3, 9, 11, 17, 19, \ldots, 737$ and $f_0 = f_s / 2^{11} \approx 2.38 \text{ Hz}$
- $N = 2048$ points per period, $M = 8$ periods
2. Nonparametric models of LTI systems

2.d Arbitrary excitations – output error problem

- Noisy output measurements, known input
- Divide the input-output records in $M$ segments

\[
\begin{align*}
    m &= 1 & 2 & \ldots & M \\
    \text{N data points} & & & & \text{N data points}
\end{align*}
\]

Leakage reduction (Schoukens et al., 2006)
- $M$ as small as possible for a given $N_T$
- $Y_D^{[m]} = \text{diff}(Y^{[m]})$, $U_D^{[m]} = \text{diff}(U^{[m]})$ (better than Hann window!)

Sample auto- and cross-power spectra

\[
\begin{align*}
    \hat{S}_{Y_D}(k) &= \frac{1}{M} \sum_{m=1}^{M} |Y_D^{[m]}(k)|^2, \\
    \hat{S}_{U_D}(k) &= \frac{1}{M} \sum_{m=1}^{M} |U_D^{[m]}(k)|^2, \\
    \hat{S}_{Y_DU_D}(k) &= \frac{1}{M} \sum_{m=1}^{M} Y_D^{[m]}(k) U_D^{[m]}(k)
\end{align*}
\]
2. Nonparametric models of LTI systems

2.d Arbitrary excitations – output error problem (cont’d)

• Least squares estimator

\[ V_{LS}(G(j\omega_{k+1/2}), Z) = \sum_{m=1}^{M} |Y_B^{[m]}(k) - G(j\omega_{k+1/2})U_B^{[m]}(k)|^2 \]

Minimise w.r.t. \( G(j\omega_{k+1/2}) \) \( \Rightarrow \)

\[ \hat{G}_{LS}(j\omega_{k+1/2}) = \frac{\hat{S}_{YDU_D}(k)}{\hat{S}_{U_D}(k)} \]

Sample variance

\[ \hat{\sigma}_Y^2(k + 1/2) = \frac{M}{2(M-1)} \left( \hat{S}_{Y_D}(k) - \left| \frac{\hat{S}_{YDU_D}(k)}{\hat{S}_{U_D}(k)} \right|^2 \right) \]

Asymptotic \((M \to \infty)\) properties LS estimate FRF for known fixed input
- consistent
- asymptotically circular complex normally distributed
- asymptotically efficient
- expected value and variance exist
- not robust w.r.t. input measurement noise
2. Nonparametric models of LTI systems

2.d Arbitrary excitations – output error problem (cont’d)

• Least squares estimator (cont’d)

  Uncertainty bounds for fixed input

  \[ \hat{G}_{LS}(j\omega_k) - G_0(j\omega_k) \in \mathcal{A}_cN_c(0, \hat{\sigma}_G^2(k)) \]

  with

  \[ \hat{\sigma}_G^2(k + 1/2) = \frac{\hat{\sigma}_Y^2(k)/M}{\hat{S}_{UD}(k)} = \frac{2\hat{\sigma}_Y^2(k + 1/2)/M}{\hat{S}_{UD}(k)} \]

  \[ \Rightarrow p\% \text{ confidence region} = \text{circle with radius} \]

  \[ \sqrt{-\log(1-p)}\hat{\sigma}_G(k) \]

  Notes:

  - decrease leakage error: \( M \) small for a given \( N_T = M \times N \)
  - decrease uncertainty FRF estimate: \( M \) large for a given \( N_T = M \times N \)
  - frequency resolution FRF estimate \( f_s/N \)
2. Nonparametric models of LTI systems

2.e Simulation example: FRF and variance estimate using arbitrary input

\[ N = 2500, M = 2 \]
2. Nonparametric models of LTI systems

2.e Simulation example: FRF estimate using arbitrary input (cont’d)

Note: variance estimate averaged over 30 frequencies
2. Nonparametric models of LTI systems

2.f Nonlinear systems – response to a periodic input

\[
\begin{align*}
\cos^2(\omega_0 t) &= \left( \frac{1}{2} (e^{j\omega_0 t} + e^{-j\omega_0 t}) \right)^2 = \frac{1}{4} (e^{j2\omega_0 t} + 2 + e^{-j2\omega_0 t}) = \frac{1}{2} (1 + \cos(2\omega_0 t))
\end{align*}
\]
2. Nonparametric models of LTI systems

2.f Nonlinear systems – response to a periodic input (cont’d)

\[
\cos^{2n}(\omega_0 t) = \frac{1}{2^n} C_n^{2n} + \frac{1}{2^{n-1}} \sum_{r=0}^{n-1} C_n^r \cos(2(n-r)\omega_0 t)
\]
2. Nonparametric models of LTI systems

2.f Nonlinear systems – response to a periodic input (cont’d)

\[ u(t) \quad \rightarrow \quad u^3 \quad \rightarrow \quad y(t) \]

\[ U(k) \quad \rightarrow \quad u^3 \quad \rightarrow \quad Y(k) \]

\[
\cos^3(\omega_0 t) = \frac{1}{8} \left( e^{j3\omega_0 t} + 3e^{j\omega_0 t} + 3e^{-j\omega_0 t} + e^{-j3\omega_0 t} \right) = \frac{1}{4} \left( 3\cos(\omega_0 t) + \cos(3\omega_0 t) \right)
\]
2. Nonparametric models of LTI systems

2.f Nonlinear systems – response to a periodic input (cont’d)

\[ u(t) \xrightarrow{\sinh(u)} y(t) \]

\[ U(k) \xrightarrow{\sinh(u)} Y(k) \]

\[
\cos^{2n+1}(\omega_0 t) = \frac{1}{2^{2n}} \sum_{r=0}^{n} C_{2n+1}^r \cos((2(n-r) + 1)\omega_0 t)
\]
2. Nonparametric models of LTI systems

2.f Nonlinear systems – response to a periodic input (cont’d)

\[ u(t) = 2 \cos(\omega_0 t + \phi_1) + 2 \cos(3 \omega_0 t + \phi_3) + 2 \cos(5 \omega_0 t + \phi_5) \]
2. Nonparametric models of LTI systems

2.f Nonlinear systems – response to a periodic input (cont’d)

\[ u(t) = 2 \cos(\omega_0 t + \phi_1) + 2 \cos(3\omega_0 t + \phi_3) + 2 \cos(5\omega_0 t + \phi_5) \]
2. Nonparametric models of LTI systems

2.f Nonlinear systems – response to a periodic input (cont’d)

\[ u(t) = 2 \cos(\omega_0 t + \phi_1) + 2 \cos(3 \omega_0 t + \phi_3) + 2 \cos(5 \omega_0 t + \phi_5) \]
2. Nonparametric models of LTI systems

2.g Nonlinear systems – best linear approximation

Class of Gaussian excitation signals with the same Riemann equivalent power spectrum:
- Gaussian noise
- periodic Gaussian noise
- random phase multisine

\[
\begin{align*}
  y(t) + y_{\text{DC}} &= 0 \\
  \mathbb{E}\{y(t)\} &= 0
\end{align*}
\]

\[
\begin{align*}
  u(t) + u_{\text{DC}} &= 0 \\
  \mathbb{E}\{u(t)\} &= 0
\end{align*}
\]

\[
u(t) = \sqrt{F} \sum_{k = -F, k \neq 0}^{F} U_k e^{j(2\pi kf_s/N)t}
\]

with

\[
F < N/2, \text{ and } F/N \text{ independent of } N
\]

\[
|U_k| \text{ user defined, and } \mathbb{E}\{e^{j\angle U_k}\} = 0
\]

power \(N^{-1}\sum_{n=0}^{N-1} u^2(nT_s)\) is independent of \(N\)
Gaussian noise and random phase multisines with the same Riemann equivalent power spectrum:
2. Nonparametric models of LTI systems

2.g Nonlinear systems – best linear approximation (cont’d)

Wiener systems:
- steady state response has same period as input (PISPO)
- saturation, discontinuities (e.g. relays) are allowed
- no chaos nor sub-harmonics
2. Nonparametric models of LTI systems

2.g Nonlinear systems – best linear approximation (cont’d)

Definition best linear approximation (BLA):

\[ G_{BLA}(j\omega) = \frac{S_{yu}(j\omega)}{S_{uu}(j\omega)} = \frac{F\{E\{y(t)u(t-\tau)\}\}}{F\{E\{u(t)u(t-\tau)\}\}} \]

where the expected value is taken w.r.t. the random realisation of \( u(t) \)
2. Nonparametric models of LTI systems

2.g Nonlinear systems – best linear approximation (cont’d)

\[ G_{\text{BLA}}(s) = G_0(s) + G_B(s) \]

\[ y(t) = y_{BLA}(t) \]

\[ Y_s(k) = Y_{\text{BLA}}(j\omega_k) \]

\[ Y(k) = Y(k) \]

\[ G_0(s) : \]
- true underlying linear system (if it exists)

\[ G_B(s) : \]
- bias contribution
- depends on input power spectrum and odd nonlinear distortions only
- smooth function of the frequency

\[ G_{\text{BLA}}(s) : \]
- best linear approximation or related linear dynamic system or LTI second order equivalent
- can be approximated very well by a rational form in \( s \)
2. Nonparametric models of LTI systems

2.g Nonlinear systems – best linear approximation (cont’d)

$y_s(t)$:
- periodic signal
- depends on input power spectrum and the even and odd nonlinearities
- zero mean
- uncorrelated with – but not independent of – $u(t)$
- not normally distributed
2. Nonparametric models of LTI systems

2.g Nonlinear systems – best linear approximation (cont’d)

\[ Y_S(k) : \]

- \[ \mathbb{E} \{ Y_S(k) \overline{U}(k) \} = 0 \Rightarrow Y_S(k) \text{ uncorrelated with } - \text{ but not independent of } - U(k) \]
- \[ \mathbb{E} \{ Y_S(k) \} = 0 \]
- \[ Y_S(k) \text{ is asymptotically uncorrelated over } k \text{ (cumulant mixing over } k) \]
- \[ \text{asymptotically normally distributed} \]
- \[ \mathbb{E} \{ |Y_S(k)|^2 \} \text{ is a smooth function of the frequency} \]

Note: expectations are taken over the different random realisations of the excitation
2. Nonparametric models of LTI systems

2.h Nonlinear systems – noisy output measurements

\[ N_Y(k) : \]
- independent of \( U_0(k) \)
- \( \mathbb{E}\{N_Y(k)\} = 0 \)
- asymptotically normally distributed
- asymptotically independent over \( k \)
- \( \mathbb{E}\{|N_Y(k)|^2\} \) is a smooth function of the frequency

Question
- distinction between \( y_s(t) \) and \( n_y(t) \)?
2. Nonparametric models of LTI systems

2.h Nonlinear systems – noisy output measurements (cont’d)

- FRF measurement using random phase multisine

\[
\begin{align*}
U_0^{[p]} &= U_0, \quad Y_s^{[p]} = Y_s, \quad N_Y^{[p]} \\
\hat{G}_{ML} &= G_{BLA} + Y_s/U_0 + \hat{N}_Y/U_0 \\
\text{var}(\hat{G}_{ML}) &= \text{var}(N_Y^{[p]})/(P|U_0|^2)
\end{align*}
\]

- FRF measurement using Gaussian noise

\[
\begin{align*}
U_0^{[p]}, \quad Y_s^{[p]}, \quad N_Y^{[p]} &\implies \\
\hat{G}_{LS} &= G_{BLA} + \hat{S}_{Y_s}U_0/\hat{S}_{U_0}U_0 + \hat{S}_{N_Y}U_0/\hat{S}_{U_0}U_0 \\
\text{var}(\hat{G}_{LS}) &= (\text{var}(N_Y^{[p]}) + \text{var}(Y_s^{[p]}))/(P\hat{S}_{U_0}U_0)
\end{align*}
\]
2. Nonparametric models of LTI systems

2.h Nonlinear systems – noisy output measurements (cont’d)

- FRF measurement using random phase multisine (cont’d)

\[
\hat{G}^{[m,p]} = \frac{Y^{[m,p]}}{U^{[m]}_0} = G_{BLA} + \frac{Y_S^{[m]}}{U^{[m]}_0} + \frac{N_Y^{[m,p]}}{U^{[m]}_0}
\]
2. Nonparametric models of LTI systems

2.h Nonlinear systems – noisy output measurements (cont’d)

- FRF measurement using random phase multisine (cont’d)

Sample mean and sample variance of each realisation

\[
\hat{G}^{[m]} = \frac{1}{P} \sum_{p=1}^{P} \hat{G}^{[m,p]} \\
\hat{\sigma}_n^{2[m]} = \frac{1}{P(P-1)} \sum_{p=1}^{P} |\hat{G}^{[m,p]} - \hat{G}^{[m]}|^2, \quad \hat{\sigma}_n^2 = \frac{1}{M^2} \sum_{m=1}^{M} \hat{\sigma}_n^{2[m]}
\]

Sample mean and sample variance of the mean values

\[
\hat{G}_{ML} = \frac{1}{M} \sum_{m=1}^{M} \hat{G}^{[m]} \\
\hat{\sigma}_{ML}^2 = \frac{1}{M(M-1)} \sum_{m=1}^{M} |\hat{G}^{[m]} - \hat{G}_{ML}|^2
\]

Properties

\[
\mathbb{E} \{ \hat{\sigma}_n^2 \} = \frac{\text{var}(N_Y)}{MP|U_0|^2}
\]

\[
\mathbb{E} \{ \hat{\sigma}_{ML}^2 \} = \frac{\text{var}(Y_s)}{M|U_0|^2} + \frac{\text{var}(N_Y)}{MP|U_0|^2}
\]

Note the asymmetric averaging of \( Y_s \) and \( N_Y \)
2. Nonparametric models of LTI systems

2.i Simulation example: FRF estimate in the presence of nonlinear distortions

Multisine
- \( M = 4, \; P = 2 \), period length \( N = 12500 \)
- \( \text{std}(u(t)) = 1 \)

Gaussian noise
- record length \( M \times P \times N \Rightarrow \) same measurement time
- segment length \( N \times P \Rightarrow \) same averaging of the stochastic NL distortions
- same Riemann equivalent power spectrum
2. Nonparametric models of LTI systems

2.i Simulation example: FRF estimate in the presence of nonlinear distortions (cont’d)

Note:
- random noise is $P = 2$ times more averaged for the multisine experiment
2. Nonparametric models of LTI systems

2.j Nonlinear systems – noisy input/output measurements

Noisy input/output measurements

\[
y(t) = y_{\text{BLA}}(t) + y_S(t) + n_y(t)
\]

\[
u(t) = u_0(t) + n_u(t)
\]

Two cases

1. reference signal \( r(t) \) is not available
2. reference signal \( r(t) \) is available
2. Nonparametric models of LTI systems

2.j Nonlinear systems – noisy input/output measurements (cont’d)

Case 1

- $M$ different realisations of $P$ periods each
- for each realisation: sample mean and sample variance input/output spectra over the periods $\Rightarrow$ FRFs + noise variance (see section 2.b, p. 37-40)
- sample mean and sample variance FRFs over the realisations (see section 2.h, p. 65)

Case 2

- $M$ different realisations of $P$ periods each
- calculate the input, output, and reference DFT spectra of each period
- projection step

$$U_R^{[m,p]} = \frac{U^{[m,p]}}{R^{[m]}}, \quad Y_R^{[m,p]} = \frac{Y^{[m,p]}}{R^{[m]}} \quad \text{or} \quad U_R^{[m,p]} = \frac{U^{[m,p]}}{e^{j\angle R^{[m]}_m}}, \quad Y_R^{[m,p]} = \frac{Y^{[m,p]}}{e^{j\angle R^{[m]}}}
$$
- variance analysis of the set of projected input/output DFT spectra

$$\hat{\sigma}_U^2 \Rightarrow \text{var}(N_U), \quad \hat{\sigma}_Y^2 \Rightarrow \text{var}(N_Y), \quad \text{var}(Y_s), \quad \hat{\sigma}^2_{Y \cdot U_R} \Rightarrow \text{covar}(N_Y, N_U)
$$
- calculate uncertainty on FRF (see section 2.b, p. 40)
2. Nonparametric models of LTI systems

2.k Nonlinear systems operating in feedback

Why is the problem important?

Feedback is often present in an experimental setup:

- either due to a control action
- or due to the interaction between generator/actuator and the dynamical system
2. Nonparametric models of LTI systems

2.k Nonlinear systems operating in feedback (cont’d)

Example interaction between generator/actuator and the dynamical system

\[ Z(s) \]
\[ i(t) \]
\[ e(t) \]
\[ v(t) \]

\[ \downarrow \]

\[ \frac{1}{Z(s)} \]

\[ r(t) = e(t) \]
\[ u(t) = i(t) \]
\[ y(t) = v(t) \]
2. Nonparametric models of LTI systems

2.k Nonlinear systems operating in feedback (cont’d)

Definition best linear approximation (BLA):

\[ G_{BLA}(j\omega) = \frac{S_{yr}(j\omega)}{S_{ur}(j\omega)} = \frac{F\{\mathbb{E}\{y(t)r(t-\tau)\}\}}{F\{\mathbb{E}\{u(t)r(t-\tau)\}\}} \]

where the expected value is taken w.r.t. the random realisation of \(r(t)\)
2. Nonparametric models of LTI systems

2.k Nonlinear systems operating in feedback (cont’d)

Does the new definition make sense?

Verify the following special cases

- nonlinear system operating in open loop
- linear system operating in a nonlinear feedback loop
2. Nonparametric models of LTI systems

2.k Nonlinear systems operating in feedback (cont’d)

Nonlinear system operating in open loop

\[
G_{BLA}(j\omega) = \frac{S_{yr}(j\omega)}{S_{ur}(j\omega)} = \frac{S_{yu}(j\omega) G_{act}^{-1}(j\omega)}{S_{uu}(j\omega) G_{act}^{-1}(j\omega)} = \frac{S_{yu}(j\omega)}{S_{uu}(j\omega)}
\]
2. Nonparametric models of LTI systems

2.k Nonlinear systems operating in feedback (cont’d)

Linear system operating in a nonlinear feedback loop

\[ G_{BLA}(j\omega) = \frac{S_{yr}(j\omega)}{S_{ur}(j\omega)} = \frac{G(j\omega)S_{ur}(j\omega)}{S_{ur}(j\omega)} = G(j\omega) \]

\[ Y_s(k) = Y(k) - G(j\omega_k)U(k) = 0 \]
2. Nonparametric models of LTI systems

2.k Nonlinear systems operating in feedback (cont’d)

General class of random excitations $r(t)$ with given joint pdf and power spectrum

$$y_s(t) = y_{s,DC} = y_{DC} - G_{BLA}(0)u_{DC}$$

$y_s(t)$:
- zero mean
- uncorrelated with – but not independent of – $r(t)$
- not normally distributed
Class of Gaussian excitation signals $r(t)$ with given Riemann equivalent power spectrum

$Y_S(k)$:

- $\mathbb{E}\{Y_S(k)\bar{R}(k)\} = 0 \Rightarrow Y_S(k)$ uncorrelated with but not independent of $R(k)$
- $\mathbb{E}\{Y_S(k)\} = 0$
- $Y_S(k)$ is asymptotically uncorrelated over $k$ (cumulant mixing over $k$)
- asymptotically normally distributed
- $\mathbb{E}\{|Y_S(k)|^2\}$ is a smooth function of the frequency

Note: expectations are taken over the different random realisations of the reference signal
2. Nonparametric models of LTI systems

2.k Nonlinear systems operating in feedback (cont’d)

Best linear approximation open loop system from reference to input-output simultaneously

\[ U(k) = G_{ru}(j\omega_k)R(k) + \tilde{U}_S(k) \]
\[ Y(k) = G_{ry}(j\omega_k)R(k) + \tilde{Y}_S(k) \]

where \( \tilde{U}_S(k), \tilde{Y}_S(k) \): uncorrelated with \( R(k) \)

Best linear approximation from input to output

\[ G_{BLA}(j\omega_k) = \frac{\mathbb{E}\{ Y(k)\bar{R}(k) \}}{\mathbb{E}\{ U(k)\bar{R}(k) \}} = \frac{G_{ry}(j\omega_k)}{G_{ru}(j\omega_k)} \]

Relationship observed input-output distortions \( \tilde{U}_S(k), \tilde{Y}_S(k) \) and output residual \( Y_S(k) \)

\[ Y_S(k) = Y(k) - G_{BLA}(j\omega_k)U(k) = \tilde{Y}_S(k) - G_{BLA}(j\omega_k)\tilde{U}_S(k) \]
2. Nonparametric models of LTI systems

2.1 Measurement example: the open loop gain of an operational amplifier

Excitation $v_g(t)$:
- odd random phase multisine with random harmonic grid
- logarithmic frequency distribution between 10 Hz and 100 kHz
- $P = 5$ periods, $M = 25$ experiments, $N = 64 \times 1024$ samples per period, $f_s = 625$ kHz
2. Nonparametric models of LTI systems

2.1 Measurement example: the open loop gain of an operational amplifier (cont’d)
2. Nonparametric models of LTI systems

2.1 Measurement example: the open loop gain of an operational amplifier (cont’d)

\[ M = 25, P = 5 \]

\[ \hat{A}_{BLA} = \frac{\text{var}(N_Y(k))}{MP|U|^2} + \frac{\text{var}(Y_S)}{M|U|^2} \]

\[ \text{Open loop gain} \]

+ fast method

+ robust method
2. Nonparametric models of LTI systems

2.1 Measurement example: the open loop gain of an operational amplifier (cont’d)

\[ M = 25, \ P = 5 \]

\[ v^- = 6.1 \text{ mVrms} \]

\[ v^- = 12.3 \text{ mVrms} \]
2. Nonparametric models of LTI systems

2.1 Measurement example: the open loop gain of an operational amplifier (cont’d)

Equivalent scheme

excited harmonics only

excited and non-excited harmonics
2. Nonparametric models of LTI systems

2.1 Measurement example: the open loop gain of an operational amplifier (cont’d)

FRF at all frequencies

Open loop gain

Amplitude (dB)

log_{10}(Frequency (Hz))

-50

0

50

excited harm.

non-excited harm.

even non-excited harm.

odd non-excited harm.

FRF

var(FRF)
2. Nonparametric models of LTI systems

2.m Measurement example: car body in white

Excitation

full random phase multisine
uniform frequency distribution in [5 Hz, 400 Hz]

\[ F = 1011 \text{ excited harmonics} \]

Robust method using the reference signal

\[ P = 2 \text{ periods} \]
\[ M = 40 \text{ realisations} \]
2. Nonparametric models of LTI systems

2.m Measurement example: car body in white

Note

linear modes: total variance = noise variance
nonlinear modes: total variance > noise variance
2. Nonparametric models of LTI systems

2.n Linear time-variant systems – examples
2. Nonparametric models of LTI systems

2.n Linear time-variant systems – examples (cont’d)

Goal: understanding/monitoring pit corrosion
Observation: impedance evolves in time
Reason: pitting
Goal: distinguish normal, ischemic and infarcted heart tissue
Observation: periodic change impedance
Reason: expansion/contraction heart
2. Nonparametric models of LTI systems

2.n Linear time-variant systems – examples arbitrary time-variation (cont’d)

- height, speed
- thermal drift
- fatigue, aging mortification
2. Nonparametric models of LTI systems

2.n Linear time-variant systems – examples periodic time-variation (cont’d)
2. Nonparametric models of LTI systems

2.0 Linear time-variant systems – introduction

The input-output relationship is given by the following convolution integral

\[ y(t) = \int_{-\infty}^{+\infty} g(t, \tau) u(\tau) d\tau \]

where \( g(t, \tau) \) is the response of the system to a Dirac impulse at time \( \tau \)
\((t = \text{response time, } \tau = \text{impulse time})\)

For causal systems \( g(t, \tau) = 0 \) for \( t < \tau \)
2. Nonparametric models of LTI systems

2.0 Linear time-variant systems – introduction (cont’d)

Define the time-variant frequency response function as

\[ G(j\omega, t) = \int_{-\infty}^{+\infty} g(t, t - \tau)e^{-j\omega \tau} d\tau \]

where \(-\infty\) is replaced by 0 for causal systems

Properties time-variant FRF (Zadeh, 1950)

- Steady state response to a periodic input \( u(t) = \sin(\omega_0 t) \)

\[ y(t) = |G(j\omega_0, t)| \sin(\omega_0 t + \angle G(j\omega_0, t)) \]

- Transient response to \( u(t) \)

\[ y(t) = L^{-1}\{G(s, t)U(s)\} \]

similar to those of the FRF of an LTI system
2. Nonparametric models of LTI systems

2.0 Linear time-variant systems – introduction (cont’d)

Definition best linear time-invariant approximation (BLTI)

\[ G_{BLTI}(j\omega) = \frac{1}{T} \int_{0}^{T} G(j\omega, t)dt \]

with \( T \)

- the observation time for arbitrary time-varying systems
- the period for periodically time-varying systems

Assumptions

- The time-variant FRF can be written as

\[ G(j\omega, t) = \sum_{p=0}^{\infty} G_p(j\omega)f_p(t) \]

with \( f_p(t) \) a complete set of basis functions over \([0, T]\) such that

\[ f_0(t) = 1 \text{ and } \frac{1}{T} \int_{0}^{T} f_p(t)dt = 0 \text{ for } p > 0 \]

- \( u(t) \) is periodic noise or a random phase multisine
- The LTI systems \( G_p(s) \) operate in steady state
2. Nonparametric models of LTI systems

2.0 Linear time-variant systems – introduction (cont’d)

Result

\[
G_{BLTI}(j\omega_k) = G_0(j\omega_k) = \frac{\mathbb{E}\{Y(k)\overline{U(k)}\}}{\mathbb{E}\{|U(k)|^2\}}
\]

where \(\mathbb{E}\{\cdot\}\) is taken w.r.t. the random realisation of \(u(t)\), is the best (in least squares sense) linear time-invariant approximation of the LTV system \(G(j\omega, t)\)

Conclusion

\[
\begin{align*}
Y_{TV}(k) &\text{ is uncorrelated with } - \text{ but not independent of } - \text{ the input } U(k)
\end{align*}
\]
2. Nonparametric models of LTI systems

2.0 Linear time-variant systems – introduction (cont’d)

Properties $G_{BLTI}(j\omega)$

- depends on $T$
- independent of the input power spectrum
- independent of the choice of the basis functions $f_{\nu}(t)$
2. Nonparametric models of LTI systems

2.0 Linear time-variant systems – introduction (cont’d)

Properties $Y_{TV}(k)$

- $\mathbb{E}\{ Y_{TV}(k) \overline{U}(k) \} = 0 \Rightarrow Y_{TV}(k)$ uncorrelated with – but not independent of – $U(k)$
- $\mathbb{E}\{ Y_{TV}(k) \} = 0$
- $Y_{TV}(k)$ is correlated over $k$
- $Y_{TV}(k)$ is not normally distributed
- $\mathbb{E}\{ |Y_{TV}(k)|^2 \}$ is a smooth function of the excited frequencies
- $u(t) \rightarrow \alpha u(t) \Rightarrow Y_{TV}(k) \rightarrow \alpha Y_{TV}(k)$ (reason: linear system!)

Note: expectations are taken over the different random realisations of the excitation
2. Nonparametric models of LTI systems

2.p Linear time-variant systems operating in feedback

Definition best linear time-invariant approximation (BLTI) via the indirect method

\[
G_{\text{BLTI}}(j\omega) = \frac{1}{T} \int_0^T G_{ry}(j\omega, t) dt = \frac{\mathbb{E}\{Y(k)R(k)\}}{\mathbb{E}\{U(k)R(k)\}}
\]

with \( T \)
- the observation time for arbitrary time-varying systems
- the period for periodically time-varying systems
where $Y_{TV}(k)$ is uncorrelated with – but not independent of – $R(k)$ (other properties similar to open loop case)
2. Nonparametric models of LTI systems

2.p Linear time-variant systems operating in feedback (cont’d)

BLTI approximation open loop system from reference to input-output simultaneously:

\[ U(k) = G_{ru}(j\omega_k)R(k) + \tilde{U}_{TV}(k) \]
\[ Y(k) = G_{ry}(j\omega_k)R(k) + \tilde{Y}_{TV}(k) \]

where \( \tilde{U}_{TV}(k), \tilde{Y}_{TV}(k) \): uncorrelated with \( R(k) \)

BLTI approximation from input to output

\[ G_{BLTI}(j\omega_k) = \frac{\mathbb{E}\{Y(k)R(k)\}}{\mathbb{E}\{U(k)R(k)\}} = \frac{G_{ry}(j\omega_k)}{G_{ru}(j\omega_k)} \]

Relationship observed input-output time-variation \( \tilde{U}_{TV}(k), \tilde{Y}_{TV}(k) \) and output residual \( Y_{TV}(k) \)

\[ Y_{TV}(k) = Y(k) - G_{BLTI}(j\omega_k)U(k) = \tilde{Y}_{TV}(k) - G_{BLTI}(j\omega_k)\tilde{U}_{TV}(k) \]
2. Nonparametric models of LTI systems

2.q Linear time-variant systems – noisy output measurements

Problem

$Y_{TV}(k)$ increases the variance of the FRF measurement

Solution

model nonparametrically the time-variation $Y_{TV}(k)$ (see Chapter 4, pp. 196-213)
2. Nonparametric models of LTI systems

2.r Measurement example: time-variant electronic circuit

\[ u(t) : \text{input, random phase multisine, 1043 harmonics in the band [228.9 Hz, 39.98 kHz]} \]

\[ y(t) : \text{output} \]

\[ p(t) : \text{gate voltage, decreases linearly from -0.775 V to -1.066 V} \]
2. Nonparametric models of LTI systems

2.r Measurement example: time-variant electronic circuit (cont’d)
2. Nonparametric models of LTI systems

2.s Summary FRF measurements

Properties disturbances

<table>
<thead>
<tr>
<th></th>
<th>Filtered white noise $N_Y(k)$</th>
<th>Stochastic nonlinear distortion $Y_S(k)$</th>
<th>Time-variation $Y_{TV}(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>zero mean value</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>relation with input</td>
<td>independent</td>
<td>uncorrelated</td>
<td>uncorrelated</td>
</tr>
<tr>
<td>asympt. normally distributed</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>asympt. uncorr. over freq.</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>smooth variance</td>
<td>yes</td>
<td>at excited freq.</td>
<td>at excited freq.</td>
</tr>
</tbody>
</table>
2. Nonparametric models of LTI systems

2.s Summary FRF measurements (cont’d)

Difficulties

• distinction noise and nonlinear distortion?

• distinction time-variation and leakage error?

Peculiarities

*Increase* input rms value

• signal-to-noise-ratio *increases*

• signal-to-time-variation-ratio *remains constant*

• signal-to-distortion-ratio *decreases (mostly) or increases (e.g. dead zone)*
2. Nonparametric models of LTI systems

2.t Multivariable systems – periodic excitations

Difficulty: from one experiment one cannot calculate the frequency response matrix

Solution: perform \( n_u \) experiments

\[
Y(k) = G(j \omega_k) U(k)
\]

with

\[
Y(k): n_y \times n_u
\]

\[
U(k): n_u \times n_u
\]

\[
U_{[p,q]}(k), Y_{[p,q]}(k): p\text{-th input/output signal of the } q\text{-th experiment}
\]

\( U(k) \) regular? \( \Rightarrow \) FRM

\[
G(j \omega_k) = Y(k) U^{-1}(k)
\]

Problem: which experiments guarantee that \( U(k) \) is regular and well conditioned?
2. Nonparametric models of LTI systems

2.t Multivariable systems – periodic excitations (cont’d)

- Guaranteed well conditioned MIMO experiments

1. $n_u$ cyclic shifts of $n_u$ multisines with interleaved frequency grid,
   for example, $n_u = 3$

$$U_1^{\nu} = U_1(k) \ U_2(k) \ U_3(k)$$

Notes:
- in theory one experiment is enough
- generators interact $\Rightarrow$ perform $n_u$ experiments
- loss in frequency resolution of a factor $n_u$
- elimination interaction generators by precompensation
2. Nonparametric models of LTI systems

2.t Multivariable systems – periodic excitations (cont’d)

- Guaranteed well conditioned MIMO experiments (cont’d)

\[ 2. \mathbf{U}(k) = \mathbf{U}(k)\mathbf{W} \text{ with } \mathbf{W} \text{ an orthogonal matrix} \]

\[ W_{[p, q]} = \frac{1}{\sqrt{n_u}} \exp\left(j\frac{2\pi(p-1)(q-1)}{n_u}\right), \quad p, q = 1, 2, \ldots, n_u \]

for example, \( n_u = 3 \)

\[ \mathbf{U}(k) = \frac{\mathbf{U}(k)}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{3}} & e^{-j\frac{2\pi}{3}} \\ 1 & e^{-j\frac{2\pi}{3}} & e^{j\frac{2\pi}{3}} \end{bmatrix} \]

Notes:
- maximal frequency resolution
- different power for each input \( \mathbf{U}(k) = \mathbf{U}(k)\text{diag}([A_1(k), A_2(k), \ldots, A_{n_u}(k)])\mathbf{W} \)
- nonlinear systems \( \mathbf{U}(k) = \text{diag}([U_1(k), U_2(k), \ldots, U_{n_u}(k)])\text{diag}([1, e^{j\phi_2(k)}, \ldots, e^{j\phi_{n_u}(k)}]) \)
  with \( \mathbb{E}\{e^{j\phi_r(k)}\} = 0, \quad r = 2, 3, \ldots, n_u, \quad k = 1, 2, \ldots, F \)
- elimination interaction generators by precompensation
2. Nonparametric models of LTI systems

2.t Multivariable systems – periodic excitations (cont’d)

Variance analysis

- over consecutive periods ⇒ disturbing noise
- over different realisations of full MIMO experiments ⇒ stochastic NL distortions + disturbing noise
2. Estimation parametric plant model with estimated nonparametric noise model

2.u MIMO experiment: identification d-axis synchronous machine
2. Estimation parametric plant model with estimated nonparametric noise model

2.u MIMO experiment: identification d-axis synchronous machine (cont’d)

\[ P = 8 \text{ periods} \]
\[ N = 65536 \]
\[ [0.1 \text{ Hz}, 230 \text{ Hz}] \]
2. Estimation parametric plant model with estimated nonparametric noise model

2.u MIMO experiment: identification d-axis synchronous machine (cont’d)

![Graphs showing impedance and noise variance](image-url)
2. Estimation parametric plant model with estimated nonparametric noise model

2.u MIMO experiment: identification d-axis synchronous machine (cont’d)

\[ \text{arg}(Z_{11}) \quad \text{arg}(Z_{12}) \quad \text{arg}(Z_{21}) \quad \text{arg}(Z_{22}) \]

\[ f (\text{Hz}) \]
2. Nonparametric models of LTI systems

2.v MIMO experiment: vibrating mechanical structure

\[ \begin{align*}
  r_1 & \rightarrow \text{Shaker} & u_1 & \rightarrow \text{Al-plate} \\
  r_2 & \rightarrow u_2 & y_1 & \rightarrow y_2
\end{align*} \]

\[ n_u = n_y = 2 \]

\( r_1, r_2 \): full random orthogonal multisines
2. Nonparametric models of LTI systems

2.v MIMO experiment: vibrating mechanical structure (cont’d)

\[ M = 25, \, P = 100 \]
2. Nonparametric models of LTI systems

2.w Multivariable systems – random excitations

Segments

\[
\begin{align*}
1 & \quad 2 & \quad \ldots & \quad M \\
\end{align*}
\]

\[M \times N \quad \text{data points}\]

FRM measurement, \(M \geq n_u\)

\[
\hat{G}(j\omega_{k+1/2}) = \hat{S}_{YDU_D}(k)\hat{S}_{UDU_D}^{-1}(k)
\]

where

\[
\hat{S}_{XY} = \frac{1}{M} \sum_{m=1}^{M} X[m]Y[m]^H
\]

Noise variance measurement, \(M \geq n_u + 1\)

\[
\hat{C}_Y(k + 1/2) = \frac{M}{2(M - n_u)}(\hat{S}_{YDU_D}(k) - \hat{S}_{YDU_D}(k)\hat{S}_{UDU_D}^{-1}(k)\hat{S}_{YDU_D}^H(k))
\]

\[
\text{Cov(vec}(\hat{G}(j\omega_{k+1/2}))) = \frac{1}{M} \hat{S}_{UDU_D}(k) \otimes \hat{C}_Y(k) = \frac{2}{M} \hat{S}_{UDU_D}(k) \otimes \hat{C}_Y(k + 1/2)
\]
2. Nonparametric models of LTI systems

2.x MIMO simulation example: discrete-time system

\[
\begin{align*}
\frac{n_y}{3} & = \frac{n_u}{2} \\
M & = n_u + 1 \\
N & = 5000
\end{align*}
\]

true value

estimate
2. Nonparametric models of LTI systems

2.x MIMO simulation example: discrete-time system (cont’d)

\[ n_y = 3 \]
\[ n_u = 2 \]
\[ M = n_u + 1 \]
\[ N = 5000 \]

---

### noise power spectrum

- **true value**
- **estimate averaged over 30 frequencies**
2. Nonparametric models of LTI systems

2.y References
3. Parametric models of LTI systems

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3. Parametric models of LTI systems

3.a Band-limited versus zero-order-hold measurement set up

Zero-order-hold (ZOH)

- input known exactly $\Rightarrow$ output error problem
- actuator is part of the discrete-time model
- perfect ZOH set up

- $G_{\text{act}}(j\omega) = 1$, $G_y(j\omega) = 1$ from DC to $\infty \Rightarrow$ absolute calibration
- $G_{\text{ZOH}}(z^{-1}) = (1 - z^{-1})Z\{L^{-1}\{G(s)/s\}\}$
3. Parametric models of LTI systems

3.a Band-limited versus zero-order-hold measurement set up (cont’d)

Band-limited (BL)

- errors-in-variables problem
- actuator is not part of the continuous-time model
- perfect BL set up

$G_u(j\omega) = G_y(j\omega)$ for $|f| < f_s/2 \Rightarrow$ relative calibration

$G_u(j\omega) = G_y(j\omega) = 0$ for $|f| \geq f_s/2$
### 3. Parametric models of LTI systems

3.a Band-limited versus zero-order-hold measurement set up (cont’d)

Exact modelling: summary

<table>
<thead>
<tr>
<th></th>
<th>BL</th>
<th>ZOH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two channel measurement</td>
<td>Single channel measurement</td>
<td></td>
</tr>
<tr>
<td>Relative calibration</td>
<td>Absolute calibration</td>
<td></td>
</tr>
<tr>
<td>No flat amplitude/linear phase required</td>
<td>Flat amplitude/linear phase required</td>
<td></td>
</tr>
<tr>
<td>Instrument bandwidth ~ $f_s/2$</td>
<td>Instrument bandwidth ~ many times $f_s$</td>
<td></td>
</tr>
<tr>
<td>Anti-alias filters required</td>
<td>Anti-alias filtering not allowed</td>
<td></td>
</tr>
<tr>
<td>Continuous-time models</td>
<td>Discrete-time models</td>
<td></td>
</tr>
</tbody>
</table>

Mixing intersample behaviour and model
- ZOH measurements and CT model
- BL measurements and DT model

- **Consequences?**
  - approximation errors $\Rightarrow$ increase model complexity
  - results not portable to set ups with different intersample behaviour
3. Parametric models of LTI systems

3.a Band-limited versus zero-order-hold measurement set up (cont’d)

- Example of mixing the intersample behaviour and the model

\[ r_{zoh}(t) \rightarrow L(s) \rightarrow u(t) \rightarrow G(s) \rightarrow y(t) \]

\[ r_{zoh}(t) \text{ is a ZOH-signal } \Rightarrow \]

\[ \frac{U(z)}{R(z)} = (1 - z^{-1})Z\{L^{-1}\{L(s)/s\}\} \text{ and } \frac{Y(z)}{R(z)} = (1 - z^{-1})Z\{L^{-1}\{L(s)G(s)/s\}\} \]

Although \( u(t) \) is not a ZOH-signal, there exists a discrete-time model that exactly relates the input samples \( u(nT_s) \) to the output samples \( y(nT_s) \)

\[ G(z^{-1}) = \frac{Y(z)}{U(z)} = \frac{Y(z)/R(z)}{U(z)/R(z)} = \frac{Z\{L^{-1}\{L(s)G(s)/s\}\}}{Z\{L^{-1}\{L(s)/s\}\}} \]

Notes:
- \( G(z^{-1}) \) also depends on \( L(s) \) (⇒ intersample behaviour \( u(t) \))
- \( G(z^{-1}) \) has (in general) a larger order than the original CT system \( G_c(s) \)
- \( G(z^{-1}) \) identified with \( L(s) \) cannot be used in a set up with a different \( L(s) \)
3. Parametric models of LTI systems

3.a Band-limited versus zero-order-hold measurement set up (cont’d)

- Properties models
  - BL(cascade) = cascade(BL)
  - ZOH(cascade) ≠ cascade(ZOH)

![Diagram](image-url)

- **ZOH(G(jω)L(jω))**
- **ZOH(G(jω))ZOH(L(jω))**
3. Parametric models of LTI systems

3.b Band-limited versus zero-order-hold signal assumption

- **BL** signal spectrum
- **ZOH** signal spectrum
3. Parametric models of LTI systems

3.c Measurement example: first order system

\[ G(s) = \frac{1}{1 + RCs} \]

\[ G_{ZOH}(z^{-1}) = \frac{(1 - e^{-\frac{T_s}{RC}})z^{-1}}{1 - e^{-\frac{T_s}{RC}}z^{-1}} \]
3. Parametric models of LTI systems

3.c Measurement example: BL set up, CT model (cont’d)

\[ G(s) = \frac{1}{b_1 s + b_0} \]
3. Parametric models of LTI systems

3.c Measurement example: ZOH set up, DT model (cont’d)

\[ G_{ZOH}(z^{-1}) = \frac{a_0 z^{-1}}{1 + b_0 z^{-1}} \]
3. Parametric models of LTI systems

3.c Measurement example: mixing the assumptions (cont’d)

\[ G_{ZOH}(z^{-1}) = \frac{a_0 z^{-1}}{1 + b_0 z^{-1}} \]

\[ G(s) = \frac{1}{b_1 s + b_0} \]
3. Parametric models of LTI systems

3.c Measurement example: mixing the assumptions (cont’d)

\[
G_{ZOH}(z) = z^{-\tau} \frac{a_2 + a_1 z^{-1} + z^{-2}}{b_2 + b_1 z^{-1} + b_0 z^{-2}}
\]

BL setup, DT model

\[
G(s) = \frac{1}{b_1 s + b_0}
\]

BL setup, CT model
3. Parametric models of LTI systems

3.c Measurement example: mixing the assumptions (cont’d)

\[ G(s) = e^{-\tau s} \frac{a_2 s^2 + a_1 s + 1}{b_2 s^2 + b_1 s + b_0} \]

ZOH setup, CT model

\[ G_{ZOH}(z^{-1}) = \frac{a_0 z^{-1}}{1 + b_0 z^{-1}} \]

ZOH setup, DT model
3. Parametric models of LTI systems

3.d Band-limited versus zero-order-hold noise models

Problem statement

Noise mostly continuous-time:
- thermal noise
- 1/f noise in semi-conductor devices
- ...

Goal:
- write noise at output as filtered white noise

Reason:
- white noise source needed to prove asymptotic properties estimators
- nonparametric noise model $N \times N$ covariance matrix with Toeplitz structure
3. Parametric models of LTI systems

3.d Band-limited versus zero-order-hold noise models (cont’d)

First trial
- \( e_c(t) = \) white CT-noise

Problem
- either infinite variance or zero power spectrum
3. Parametric models of LTI systems

3.d Band-limited versus zero-order-hold noise models (cont’d)

Assumptions ZOH noise model
- $e_c(t)$ is piecewise constant where $e(m) = e_c(mT_s)$ is white DT-noise
- no anti-alias filter

At sampling instances

Note: if $e_c(t)$ is a Wiener stochastic process (integrated CT white noise) ⇒ exact DT model
3. Parametric models of LTI systems

3.d Band-limited versus zero-order-hold noise models (cont’d)

Assumptions BL noise model

- \( e_c(t) = \) band-limited white CT-noise (Åström, 1970)
- \( e_c(t) \) is Gaussian distributed
- band-limited measurement setup
- \( f_B \geq f_s/2 \)
3. Parametric models of LTI systems

3.d Band-limited versus zero-order-hold noise models (cont’d)

Proof

\[ S_{\eta}(f) = |H(j2\pi f)|^2 |AA(j2\pi f)|^2 S_{\varepsilon}(f) = |H(j2\pi f)|^2 S_{\varepsilon}(f) \]

Notes:

1. \( R_{\varepsilon}(\tau) = F^{-1}\{S_{\varepsilon}(f)\} = \sigma^2 \text{sinc}(\pi f_{s} \tau) \)
2. \( R_{\varepsilon}(nT_s) = 0 \) for \( n \neq 0 \) ⇒ \( e(n) = \varepsilon(nT_s) \) is independent, normally distributed
3. Parametric models of LTI systems

3.d Band-limited versus zero-order-hold noise models (cont’d)

Relaxed assumption on the driving CT noise source $e_c(t)$

- $S_{e_c}(f) = \text{is constant in the band } |f| < f_B$
- $S_{e_c}(f) = O(f^{-(1+\delta)})$ for $|f| \geq f_B$ and $\delta > 0$

where $O(f^{-(1+\delta)})$ is the weakest decay such that $\text{var}(e_c(t)) = \int_{-\infty}^{+\infty} S_{e_c}(f) df < \infty$
3. Parametric models of LTI systems

3.d Band-limited versus zero-order-hold noise models (cont’d)

Generalisation

• Diffusion systems (Pintelon et al., 2005)
  - plant model often rational form in $\sqrt{s}$: $G(\sqrt{s})$
  - hence also true for the process noise model: $H(\sqrt{s})$

• 1/f noise

$$H(s) = \frac{1}{\sqrt{s}}$$
3. Parametric models of LTI systems

3.d Band-limited versus zero-order-hold noise models (cont’d)

Summary

<table>
<thead>
<tr>
<th>driving noise source</th>
<th>noise model</th>
<th>application</th>
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<td>• ZOH DT white noise</td>
<td>DT</td>
<td>prediction, control</td>
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<td>(piecewise constant !?)</td>
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<td>• Wiener stochastic process</td>
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<tr>
<td>(integrated CT white noise !?)</td>
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<td>• BL CT white noise</td>
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</table>
3. Parametric models of LTI systems

3.e Lumped versus distributed LTI systems

Lumped system

\[ LC \frac{d^2 y(t)}{dt^2} + y(t) = u(t) \]

initial conditions: \( y(0) = 0, \left. \frac{dy(t)}{dt} \right|_{t=0} = 0 \)

transfer function

\[ G(s) = \frac{Y(s)}{U(s)} = \frac{1}{1 + LCs^2} \]

poles

\[ s = \pm j/\sqrt{LC} \]
3. Parametric models of LTI systems

3.e Lumped versus distributed LTI systems (cont’d)

Distributed system

\[
\frac{\partial^2 y(x, t)}{\partial t^2} = \frac{E \partial^2 y(x, t)}{\rho \partial x^2}
\]

initial conditions: \( y(x, 0) = 0, \frac{\partial y(x, t)}{\partial t} \bigg|_{t=0} = 0 \)

boundary conditions: \( y(0, t) = 0, \frac{\partial y(x, t)}{\partial x} \bigg|_{x=l} = \frac{u(t)}{E} \)

transfer function

\[
G(s) = \frac{Y(l, s)}{U(s)} = \frac{l \tanh(\tau s)}{E \tau s} = \frac{l}{E} \sum_{k=0}^{\infty} \frac{2}{(\tau s)^2 + (\pi(2k+1)/2)^2} \quad \text{with} \quad \tau = \sqrt{\frac{\rho l^2}{E}}
\]

poles

\[
s = \pm (2k + 1) \frac{\pi}{2\tau}, \quad k = 1, 2, \ldots
\]

Note

\[
\left| 2/((\tau s)^2 + (\pi(2k+1)/2)^2) \right|_{s=j\omega}
\]

has limited active frequency range \( \Rightarrow \) rational approximation in given frequency band
3. Parametric models of LTI systems

3.f Lumped systems

- Continuous-time (CT)
  General
  \[
  \sum_{n=0}^{n_a} a_n y^{(n)}(t) = \sum_{m=0}^{n_b} b_m u^{(m)}(t) \Rightarrow G(s, \theta) = \frac{B(s, \theta)}{A(s, \theta)} = \frac{\sum_{r=0}^{n_b} b_r s^r}{\sum_{r=0}^{n_a} a_r s^r}
  \]

  Sometimes (diffusion phenomena such as heat or mass transfer)
  \[
  \sum_{n=0}^{n_a} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{n_b} b_m \frac{d^m u(t)}{dt^m} \Rightarrow G(\sqrt{s}, \theta) = \frac{B(\sqrt{s}, \theta)}{A(\sqrt{s}, \theta)} = \frac{\sum_{m=0}^{n_b} b_m s^{m/2}}{\sum_{n=0}^{n_a} a_n s^{n/2}}
  \]

- Discrete-time (DT)
  \[
  \sum_{n=0}^{n_a} a_n y(t-n) = \sum_{m=0}^{n_b} b_m u(t-m) \Rightarrow G(z^{-1}, \theta) = \frac{B(z^{-1}, \theta)}{A(z^{-1}, \theta)} = \frac{\sum_{r=0}^{n_b} b_r z^{-r}}{\sum_{r=0}^{n_a} a_r z^{-r}}
  \]

- Transient behaviour
  - differential and difference equations: exponential decay
  - fractional differential equation: \(O(t^{-3/2})\) decay
3. Parametric models of LTI systems

3.f Lumped systems (cont’d)

Summary

\[ \begin{align*}
G(\Omega, \theta) &= \frac{B(\Omega, \theta)}{A(\Omega, \theta)} = \frac{\sum_{r=0}^{n_b} b_r \Omega^r}{\sum_{r=0}^{n_a} a_r \Omega^r} \\
& \quad \text{with } \Omega = \begin{cases} 
  s & \text{continuous-time} \\
  \sqrt{s} & \text{diffusion} \\
  z^{-1} & \text{discrete-time}
\end{cases}
\end{align*} \]
3. Parametric models of LTI systems

3.g Relation between input/output DFT spectra - plant model only

- Input/output DFT spectra
  \[
  U(k) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} u(tT_s)z_k^{-t},
  Y(k) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} y(tT_s)z_k^{-t}
  \]
  with \( z_k = e^{j2\pi k/N} \)

- Periodic signals
  \[
  Y(k) = G(\Omega_k, \theta) U(k)
  \]
  if
  - steady state response
  - integer number of periods are observed
3. Parametric models of LTI systems

3.3 Relation between input/output DFT spectra - plant model only (cont’d)

- Arbitrary signals

\[ Y(k) = G(\Omega_k, \theta)U(k) + T_G(\Omega_k, \theta) + \delta(\Omega_k) \]

with

\[ G(\Omega, \theta) = \frac{B(\Omega, \theta)}{A(\Omega, \theta)} = \frac{\sum_{r=0}^{n_b} b_r \Omega^r}{\sum_{r=0}^{n_a} a_r \Omega^r}, \quad T_G(\Omega, \theta) = \frac{I_G(\Omega, \theta)}{A(\Omega, \theta)} = \frac{\sum_{r=0}^{i_{g_r}} i_{g_r} \Omega^r}{\sum_{r=0}^{n_a} a_r \Omega^r} \]

where

\[ i_{g_r} = \text{function of the difference between the initial and final conditions} \]

\[ n_{i_g} = \max(n_a, n_b) - 1 \text{ for } \Omega = z^{-1} \text{ and } n_{i_g} = \max(n_a, n_b) \text{ for } \Omega = s, \sqrt{s} \]

\[ |U(k)| = O(N^0), \quad |T_G(\Omega_k, \theta)| = O(N^{-1/2}) \]

\[ \delta(\Omega_k) = \begin{cases} 0 & \Omega = z^{-1} \\ \text{residual alias error } O(N^{-1/2}) & \Omega = s, \sqrt{s} \end{cases} \]

Notes:

- \( T_G(\Omega, \theta) = \delta(\Omega_k) = 0 \) when the initial conditions equal the final conditions
- reduce \( \delta(\Omega_k) \) for \( \Omega = s, \sqrt{s} \) by choice \( n_{i_g} \geq \max(n_a, n_b) \)
3. Parametric models of LTI systems

3.g Relation between input/output DFT spectra - plant model only (cont’d)

- Summary

$$Y(k) = G(\Omega_k, \theta) U(k) + T_G(\Omega_k, \theta)$$

with

$$G(\Omega, \theta) = \frac{B(\Omega, \theta)}{A(\Omega, \theta)} = \frac{\sum_{r=0}^{n_b} b_r \Omega^r}{\sum_{r=0}^{n_a} a_r \Omega^r}, \quad T_G(\Omega, \theta) = \frac{I_G(\Omega, \theta)}{A(\Omega, \theta)} = \frac{\sum_{r=0}^{n_{ig}} i_{gr} \Omega^r}{\sum_{r=0}^{n_a} a_r \Omega^r}$$

where

$$\Omega = \begin{cases} z^{-1} & n_{ig} = \max(n_a, n_b) - 1 \\ s, \sqrt{s} & n_{ig} > \max(n_a, n_b) - 1 \end{cases}$$

and

$$T_G(\Omega_k, \theta) = \begin{cases} 0 & \text{for periodic and time-limited signals} \\ O(N^{-1/2}) & \text{arbitrary signals} \end{cases}$$
3. Parametric models of LTI systems

3.h Simulation example: 4th order discrete-time Butterworth filter

Cut off frequency $f_s/8$
3. Parametric models of LTI systems

3.h Simulation example: 4th order discrete-time Butterworth filter (cont’d)

Note:
- random behaviour error \( T_G(z_k^{-1}, \theta)/U(k) \) on FRF due to random behaviour \( U(k) \)
3. Parametric models of LTI systems

3.h Simulation example: 4th order discrete-time Butterworth filter (cont’d)

difference true value and model

error (dB)

without transient

with transient
3. Parametric models of LTI systems

3.h Simulation example: 4th order discrete-time Butterworth filter (cont’d)

impulse response $T_G(z^{-1}) = \frac{I_G(z^{-1})}{A(z^{-1})}$
3. Parametric models of LTI systems

3.i Simulation example: 2nd order continuous-time system

![Graph showing FRF and error models with transient and without transient](image-url)

- **true FRF**
- **error model without transient**
- **error estimate with transient**

$n_{i_s} = 10$
3. Parametric models of LTI systems

3.j Measurement example: octave bandpass filter

Plant model $n_a = 6, n_b = 4$

```
\begin{align*}
\text{measured FRF} & \quad \text{(multisine excit.)} \\
\text{error model without transient} & \quad \text{(arbitr. excit.)} \\
\text{error model with transient} n_i = 6 & \quad \text{(arbitr. excit.)} \\
\text{error model} & \quad \text{(multisine excit.)}
\end{align*}
```
3. Parametric models of LTI systems

3.k Measurement example: reduction of iron

$\sqrt{s}$-domain model with transient term
$n_a = n_b = 3$
$n_{i_g} = 5$

$\sqrt{s}$-domain model without transient term
$n_a = n_b = 3$
3. Parametric models of LTI systems

3.1 Relation between input/output DFT spectra - plant and noise model

\[
Y(k) = G(\Omega, \theta)U(k) + T_G(\Omega, \theta) + H(\Omega, \theta)E(k) + T_H(\Omega, \theta)
\]

where

\[
G(\Omega, \theta) = \frac{B(\Omega, \theta)}{A(\Omega, \theta)}, \quad T_G(\Omega, \theta) = \frac{L_G(\Omega, \theta)}{A(\Omega, \theta)}, \quad H(\Omega, \theta) = \frac{C(\Omega, \theta)}{D(\Omega, \theta)}, \quad T_H(\Omega, \theta) = \frac{L_H(\Omega, \theta)}{D(\Omega, \theta)}
\]
3. Parametric models of LTI systems

3.m Plant and noise model structures

• Frequency domain

\[ Y(k) = G(\Omega_k, \theta)U(k) + H(\Omega_k, \theta)E(k) + T_G(\Omega_k, \theta) + T_H(\Omega_k, \theta) \]

Model structures

ARX: \( C = 1, D = A, T_H = 0 \)

ARMAX: \( D = A, T_H = 0, n_i = \max(n_a, n_b, n_c) - 1 \) or \( n_i > \max(n_a, n_b, n_c) - 1 \)

ARMA: \( G = 0, T_G = 0 \)

BJ: \( D \neq A \)

Notes:

- AR = autoregressive
- MA = moving average
- X = exogenous input
- BJ = Box-Jenkins
- hybrid BJ = CT plant + DT noise model
- AR or MA or ARMA parametric noise modelling = time series analysis or spectral estimation (DT and CT)
3. Parametric models of LTI systems

3.m Plant and noise model structures (cont’d)

• DT domain

\[ y(t) = G(q, \theta)u(t) + H(q, \theta)e(t) \]

with \( q = z^{-1} \) the backward shift operator: \( qx(t) = x(t - 1) \)

and where

\[ G(q)x(t) = (\sum_{r=0}^{\infty} g(r)q^r)x(t) = \sum_{r=0}^{\infty} g(r)x(t - r) \]

• CT domain

\[ y(t) = G(p, \theta)u(t) + H(p, \theta)e(t) \]

with \( p = \frac{d}{dt} \) the derivative operator: \( px(t) = \frac{dx(t)}{dt} \)

and where

\[ G(p)x(t) = \int_{0}^{\infty} g(\tau)x(t - \tau)d\tau \]
3. Parametric models of LTI systems

3.n Parametrizations LTI systems

Plant model

- Rational form

\[ G(\Omega, \theta) = \frac{B(\Omega, \theta)}{A(\Omega, \theta)} = \frac{\sum_{r=0}^{n_b} b_r \Omega^r}{\sum_{r=0}^{n_a} a_r \Omega^r} \quad \text{with} \quad \theta^r = [a_0 a_1 \ldots a_{n_a} b_0 b_1 \ldots b_{n_b}] \]

- Partial fraction expansion

\[ G(\Omega, \theta) = \sum_{r=-p}^{p} \frac{L_r}{\Omega - \lambda_r} + \sum_{r=1}^{q} \frac{S_r}{\Omega - \sigma_r} + W(\Omega, w) \quad \text{for} \quad \Omega = s, \sqrt{s} \]

\[ G(z^{-1}, \theta) = \sum_{r=-p}^{p} \frac{L_r z^{-1}}{1 - \lambda_r z^{-1}} + \sum_{r=1}^{q} \frac{S_r z^{-1}}{1 - \sigma_r z^{-1}} + W(z^{-1}, w) \]

with

\[ \theta^r = [\sigma_1 \ldots \sigma_q \Re(\lambda_1) \Im(\lambda_1) \ldots \Re(\lambda_p) \Im(\lambda_p) \ldots \]

\[ S_1 \ldots S_q \Re(L_1) \Im(L_1) \ldots \Re(L_p) \Im(L_p) w_0 \ldots w_{n_w} ] \]
3. Parametric models of LTI systems

3.n Parametrizations LTI systems (cont’d)

Plant model (cont’d)

• State space representation for proper transfer functions ($n_b \leq n_a$)

\[
G(s, \theta) = C(s I_{n_a} - A)^{-1} B + D
\]
\[
G(z^{-1}, \theta) = z^{-1} C(I_{n_a} - z^{-1} A)^{-1} B + D
\]

with

\[
\theta^T = [ \text{vec}^T(A) \ B^T \ C \ D ]
\]

• Pole/zero representation

\[
G(\Omega, \theta) = K \frac{\prod_{r=1}^{n_b} (\Omega - \zeta_r)}{\prod_{r=1}^{n_a} (\Omega - \lambda_r)}
\]

disadvantage: ill conditioned for multiple poles/zeroes

• Systems with time delay

\[
G(\Omega, \theta) = e^{-\tau \Omega} \frac{B(\Omega, \theta)}{A(\Omega, \theta)}
\]
\[
G(z^{-1}, \theta) = z^{-\tau/T_s} \frac{B(z^{-1}, \theta)}{A(z^{-1}, \theta)}
\]
3. Parametric models of LTI systems

3.n Parametrizations LTI systems (cont’d)

Noise model

   Similar as plant model

Note

   Starting value problem parametrizations (see Chapter 5)

       - rational form ⇒ linear least squares
       - state space ⇒ subspace methods
       - other parametrizations ⇒ via rational form + back transformation
3. Parametric models of LTI systems

3.0 Multivariable systems

- Left matrix fraction (LMF) description
  \[ G(\Omega, \theta) = A^{-1}(\Omega, \theta)B(\Omega, \theta) \text{ with } A \ n_y \times n_y, \text{ and } B \ n_y \times n_u \]

- Right matrix fraction (RMF) description
  \[ G(\Omega, \theta) = B(\Omega, \theta)A^{-1}(\Omega, \theta) \text{ with } A \ n_u \times n_u, \text{ and } B \ n_y \times n_u \]

- Common denominator
  \[ G(\Omega, \theta) = B(\Omega, \theta)/A(\Omega, \theta) \text{ with } A \ 1 \times 1, \text{ and } B \ n_y \times n_u \]

- State space
  same form as SISO with \( A \ n \times n, \ B \ n \times n_u, \ C \ n_y \times n, \text{ and } D \ n_y \times n_u \)

- Partial fraction expansion
  \( L_r, S_r \) are \( n_y \times n_u \) residue matrices

- Notes:
  - advantage LMF: \( Y(k) = G(\Omega, \theta)U(k) \Rightarrow A(\Omega, \theta)Y(k) = B(\Omega, \theta)U(k) \)
  - common denominator: rank residue matrices cannot be imposed
    (\( \Rightarrow \) too many parameters if residue matrices are not of full rank)
  - LMF, RMF, and state space: rank one residue matrices
    \( (F \ n \times n \Rightarrow F^{-1} = \text{adj}(F)/\text{det}(F) \text{ with } \text{det}(\text{adj}(F)) = (\text{det}(F))^{n-1}) \)
3. Parametric models of LTI systems

3.p References
4. Improved nonparametric models for LTI systems

4.a Arbitrary excitations — generalised output error
4.b MIMO simulation example: comparison LPM and SA
4.c Measurement example: nonlinear electrical circuit
4.d Arbitrary excitations — errors-in-variables
4.e Measurement example: flexural vibrations of a steel beam
4.f Periodic excitations — steady state
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4. Improved nonparametric models for LTI systems

4.a Arbitrary excitations — generalised output error

\[ e(t) \]
\[ H(\Omega) \]
\[ v(t) \]
\[ Y(k) = G(\Omega_k)U(k) + T(\Omega_k) + V(k) \]
\[ V(k) = H(\Omega_k)E(k) \]
\[ T(\Omega_k) = T_H(\Omega_k) + T_G(\Omega_k) \]

Goal: starting from \( U(k) \) and \( Y(k) \), obtain a nonparametric estimate of

- the frequency response function \( G(\Omega) \)
- the noise covariance matrix \( C_V(k) = \text{Cov}(V(k)) \)

Difficulties: the presence of

- \( T(\Omega_k) \) for estimating \( G(\Omega_k) \)
- \( T(\Omega_k) \) and \( G(\Omega_k)U(k) \) for estimating \( C_V(k) \)

Property:

- \( G(\Omega) \), \( H(\Omega) \), and \( T(\Omega) \) are smooth functions of the frequency
4. Improved nonparametric models for LTI systems

4.a Arbitrary excitations — generalised output error (cont’d)

Basic idea solution: local polynomial approximation of $G(\Omega)$ and $T(\Omega)$

$$Y(k + r) = G(\Omega_{k+r}) U(k + r) + T(\Omega_{k+r}) + V(k + r)$$

with Taylor series expansion

$$G(\Omega_{k+r}) = G(\Omega_k) + \sum_{s=1}^{R} g_s(k) r^s + O((r/N)^{R+1})$$

$$T(\Omega_{k+r}) = T(\Omega_k) + \sum_{s=1}^{R} t_s(k) r^s + N^{-1/2} O((r/N)^{R+1})$$

and

$$r = -n, -n + 1, \ldots, -1, 0, 1, \ldots, n - 1, n$$

$$\Theta = \begin{bmatrix} G(\Omega_k) & g_1(k) & g_2(k) & \ldots & g_R(k) & T(\Omega_k) & t_1(k) & t_2(k) & \ldots & t_R(k) \end{bmatrix}$$
4. Improved nonparametric models for LTI systems

4.a Arbitrary excitations — generalised output error (cont’d)

Local polynomial model in the band $[k - n, k + n]$:

$$Y_n = \Theta K_n + V_n$$

where

$$X_n = \begin{bmatrix} X(k-n) & X(k-n+1) & \ldots & X(k) & \ldots & X(k+n) \end{bmatrix}$$

and

$$K(k+r) = \begin{bmatrix} K_1(r) \otimes U(k+r) \end{bmatrix} \text{ with } K_1(r) = \begin{bmatrix} 1 \\ r \\ \vdots \\ r^R \end{bmatrix}$$

$$Y(k) = G(\Omega_k)U(k) + T(\Omega_k) + V(k)$$

$$V(k) = H(\Omega_k)E(k)$$

$$T(\Omega_k) = T_H(\Omega_k) + T_G(\Omega_k)$$
4. Improved nonparametric models for LTI systems

4.a Arbitrary excitations — generalised output error (cont’d)

\[ Y(k) = G(\Omega_k)U(k) + T(\Omega_k) + V(k) \]
\[ V(k) = H(\Omega_k)E(k) \]
\[ T(\Omega_k) = T_H(\Omega_k) + T_G(\Omega_k) \]

Local polynomial model

\[ Y_n = \Theta K_n + V_n \]

Linear least squares estimate

\[ \hat{\Theta} = Y_n K_n^H (K_n K_n^H)^{-1} \Rightarrow \hat{G}(\Omega_k) \]

Residual linear least squares estimate

\[ \hat{V}_n = Y_n - \hat{\Theta} K_n \Rightarrow \hat{C}_V(k) = \frac{1}{q} \hat{V}_n \hat{V}_n^H \]

with \( S = K_n^H (K_n K_n^H)^{-1} \left[ \begin{array}{c} I_{n_u} \\ 0 \end{array} \right] \)

\[ \Rightarrow \text{Cov}(\text{vec}(\hat{G}(\Omega_k))) = S^{H} S \otimes \hat{C}_V(k) \]

and \( q = 2n + 1 - (R + 1)(n_u + 1) \) the degrees of freedom \( dof \) of the residuals
4. Improved nonparametric models for LTI systems

4.a Arbitrary excitations — generalised output error (cont’d)

Repeat the previous procedure at frequency $k$ for all the other frequencies $k + 1$, $k + 2$, …

Consequence

- the FRF $\hat{G}(\Omega_k)$ and noise covariance $\hat{C}_V(k)$ estimates are correlated over the frequency
- the correlation length is $\pm 2n$
4. Improved nonparametric models for LTI systems

4.a Arbitrary excitations — generalised output error (cont’d)

Bias error local polynomial method (LPM)

\[
\mathbb{E}\{\hat{G}(\Omega_k)\} = G(\Omega_k) + G^{(R+1)}(\Omega_k)O_{\text{intG}}((n/N)^{(R+1)}) + O_{\text{leakG}}((n/N)^{(R+2)})
\]

\[
\mathbb{E}\{\hat{C}_V(k)\} = C_V(k) + C_V^{(1)}(k)O_{\text{intH}}(n/N) + G^{(R+1)}(\Omega_k)O_{\text{intG}}((n/N)^{2(R+1)})(G^{(R+1)}(\Omega_k))^H
\]

Bias error spectral analysis approach

\[
\mathbb{E}\{\hat{G}^{\text{win}}(\Omega_k)\} = G(\Omega_k) + (\alpha^{\text{win}}G)^{(1)}(\Omega_k)O_{\text{intG}}((M/N)^2) + O_{\text{leakG}}((M/N)^2)
\]

\[
\mathbb{E}\{\hat{C}_V^{\text{win}}(k)\} = C_V(k) + \frac{\alpha^{\text{win}}}{2}C_V^{(2)}(k)O_{\text{intH}}((M/N)^2) + \frac{\alpha^{\text{win}}}{2}G^{(1)}(\Omega_k)O_{\text{intG}}((M/N)^2)(G^{(1)}(\Omega_k))^H
\]

with \(\alpha^{\text{win}} = 1/4\) for \(\text{win} = \text{diff, half sine}\); and \(\alpha^{\text{win}} = 1/3\) for \(\text{win} = \text{hann}\)
4. Improved nonparametric models for LTI systems

4.a Arbitrary excitations — generalised output error (cont’d)

Local polynomial approach

- smaller bias error
- smaller uncertainty
- larger frequency resolution
- larger calculation time
- correlation length ±2n

Spectral analysis method

- larger bias error
- larger uncertainty
- smaller frequency resolution
- smaller calculation time
- correlation length 0 (rect), ±1 (diff, half sine), or ±2 (hanning)

\[ Y(k) = G(\Omega_k)U(k) + T(\Omega_k) + V(k) \]
\[ V(k) = H(\Omega_k)E(k) \]
\[ T(\Omega_k) = T_H(\Omega_k) + T_G(\Omega_k) \]
4. Improved nonparametric models for LTI systems

4. b MIMO simulation example: comparison LPM and SA

LPM and SA (hanning window): same \( \text{dof} \) noise covariance estimates
4. Improved nonparametric models for LTI systems

4.b MIMO simulation example: comparison LPM and SA (cont’d)

LPM and SA (hanning window): same $dof$ noise covariance estimates
4. Improved nonparametric models for LTI systems

4.c Measurement example: nonlinear electrical circuit

White noise excitation with bandwidth of 200 Hz

10400 samples at $f_s = 20 \text{ MHz} / 2^{15} \approx 600 \text{ Hz}$:

- split in 20 blocks of $N = 520$ samples each
- spectral analysis and local polynomial estimates on each block of $N = 520$ samples
- sample means and sample variances over the 20 blocks
4. Improved nonparametric models for LTI systems

4.c Measurement example: nonlinear electrical circuit (cont’d)

![Graphs showing FRF and noise variance for different methods](image)

- full lines: FRF
- dashed lines: std FRF
- full lines: noise var.
- dashed lines: std noise var.
4. Improved nonparametric models for LTI systems

4.d Arbitrary excitations — errors-in-variables

**Indirect method** for measuring the FRF (two possible cases):

- *Linear* plant and *(non)linear* actuator and controller: \[ G_0(j\omega_k) = S_{yr}(j\omega_k)S_{ur}^{-1}(j\omega_k) \]

- *Nonlinear* plant and *(non)linear* actuator and controller: \[ G_{\text{BLA}}(j\omega_k) = S_{yr}(j\omega_k)S_{ur}^{-1}(j\omega_k) \]

Solution:

- Reference signal \( r(t) \) needed + modeling from \( r(t) \) to \( z(t) = [y^T(t) \ u^T(t)]^T \)
4. Improved nonparametric models for LTI systems

4.d Arbitrary excitations — errors-in-variables (cont’d)

Modeling from reference to input and output:

\[
Z(k) = \begin{bmatrix} G_{ry}(\Omega_k) \\ G_{ru}(\Omega_k) \end{bmatrix} R(k) + \begin{bmatrix} T_y(\Omega_k) \\ T_U(\Omega_k) \end{bmatrix} + V_Z(k) \quad \text{with} \quad Z(k) = \begin{bmatrix} Y(k) \\ U(k) \end{bmatrix}
\]

\[
\Rightarrow \hat{G}(\Omega_k) = \hat{G}_{ry}(\Omega_k)\hat{G}_{ru}^{-1}(\Omega_k)
\]

Note: applicable to both spectral analysis and local polynomial methods
4. Improved nonparametric models for LTI systems

4.e Measurement example: flexural vibrations of a steel beam

Beam parameters: length 61 cm, height 2.47 cm, width 4.93 mm, density 7800 kg/m³

Excitation in the band [0 Hz, 6 kHz]:

- random binary sequence
- random phase multisine

\[ N = 50 \times 1024 \] samples of the reference, force (input), and acceleration (output) signals at

\[ f_s = 10 \text{ MHz}/2^9 \approx 19.53 \text{ kHz} : \]

- spectral analysis and local polynomial estimates
- generalised output error and errors-in-variables estimates
4. Improved nonparametric models for LTI systems

4.e Measurement example: flexural vibrations of a steel beam (cont’d)

random binary sequence

OE-LPM

EIV-LPM

FRF (dB)

Frequency (kHz)

FRF (dB)

Frequency (kHz)

FRF (dB)

Frequency (kHz)

FRF (dB)

Frequency (kHz)
4. Improved nonparametric models for LTI systems

4.e Measurement example: flexural vibrations of a steel beam (cont’d)

![Graphs showing FRF and std(FRF) for different methods: local polynomial approach, multisine, and spectral analysis.](image)
4. Improved nonparametric models for LTI systems

4.f Periodic excitations — steady state

Two possible cases:

- **Linear** plant and *(non)linear* actuator and controller: \( G_0(j\omega_k) = S_{yr}(j\omega_k)S_{ur}^{-1}(j\omega_k) \)

- **Nonlinear** plant and *(non)linear* actuator and controller: \( G_{BLA}(j\omega_k) = S_{yr}(j\omega_k)S_{ur}^{-1}(j\omega_k) \)

Required signals:

- \( r(t) \) (for multi-input and/or nonlinear plant only), \( u(t) \) and \( y(t) \)
4. Improved nonparametric models for LTI systems

4.f Periodic excitations — steady state (cont’d)

\[ Z(k) = Z_0(k) + H_Z(\Omega_k)E_Z(k) + T_{H_Z}(\Omega_k) \text{ with } Z(k) = \begin{bmatrix} Y(k) \\ U(k) \end{bmatrix} \]

Noise transients (leakage) errors \( T_{H_Z}(\Omega_k) \),

- Smooth functions of the frequencies
- Increase the variability of the FRF measurement (but introduce no bias!)
- Introduce a correlation between consecutive signal periods
- Important for lightly damped systems (e.g. mechanical vibrating structures)
4. Improved nonparametric models for LTI systems

4.f Periodic excitations — steady state (cont’d)

DFT spectrum of \( P = 2 \) consecutive periods

\[
Z(k) = Z_0(k) + H_Z(\Omega_k)E_Z(k) + T_{Hz}(\Omega_k)
\]
4. Improved nonparametric models for LTI systems

4.f Periodic excitations — steady state (cont’d)

2-step procedure

step 1: non-excited frequencies \( Z(k) = H_Z(\Omega_k)E_Z(k) + T_{H_Z}(\Omega_k) \)
   - estimate the noise transient term
   - estimate the noise covariance from the residuals

step 2: excited frequencies \( Z(k) = Z_0(k) + H_Z(\Omega_k)E_Z(k) + T_{H_Z}(\Omega_k) \)
   - remove estimated noise transient from the excited lines
   - estimate the plant FRF
   - estimate the total covariance from the residuals
   - difference total and noise covariance = covariance nonlinear distortions
4. Improved nonparametric models for LTI systems

4.f Periodic excitations — steady state (cont’d): STEP 1

Basic idea solution: local polynomial expansion $T_{Hz}(\Omega)$ at the non-excited DFT lines

$$Z(kP \pm m_i) = H_Z(\Omega_{kP \pm m_i})E_Z(kP \pm m_i) + T_{Hz}(\Omega_{kP \pm m_i})$$

with Taylor series expansion for $i = 1, 2, \ldots, n$ ($kP$ is not used!)

$$T_{Hz}(\Omega_{kP \pm m_i}) = T_{Hz}(\Omega_{kP}) + \sum_{r=1}^{R} t_r(k)(\pm m_i)^r + \frac{1}{\sqrt{PN}}O\left(\left(\frac{m_i}{PN}\right)^{(R+1)}\right)$$

and

$$\Theta = \left[ T_{Hz}(\Omega_k) ~ t_1(k) ~ t_2(k) ~ \ldots ~ t_R(k) \right]$$
4. Improved nonparametric models for LTI systems

4.f Periodic excitations — steady state (cont’d): \textit{STEP 1}

Local polynomial model in the band $kP \pm m_i, \ i = 1, 2, \ldots, n$

$$Z_n = \Theta K_n + V_n$$

where

$$X_n = \begin{bmatrix} X(kP - m_n) & \ldots & X(kP - m_1) & X(kP + m_1) & \ldots & X(kP + m_n) \end{bmatrix}$$

and

$$K(kP \pm m_i) = \begin{bmatrix} 1 & (\pm m_i) & \ldots & (\pm m_i)^R \end{bmatrix}^T$$
4. Improved nonparametric models for LTI systems

4.4 Periodic excitations — steady state (cont’d): STEP 1

Local polynomial model

\[ Z_n = \Theta K_n + V_n \]

Linear least squares estimate

\[ \hat{\Theta} = Z_n K_n^H (K_n K_n^H)^{-1} \Rightarrow \hat{T}_{Hz} (\Omega_{kP}) \]

Residual linear least squares estimate

\[ \hat{V}_n = Z_n - \hat{\Theta} K_n \Rightarrow \hat{C}_V(k) = \frac{1}{q} \hat{V}_n \hat{V}_n^H \]

with \( q = 2n - (R + 1) \) the degrees of freedom \( dof \) of the residuals
4. Improved nonparametric models for LTI systems

4.f Periodic excitations — steady state (cont’d): STEP 2

At the excited DFT lines $kP$

$$Z(kP) = Z_0(kP) + V(kP) + T_{Hz}(\Omega_{kP})$$

Sample mean

$$\hat{Z}(kP) = Z(kP) - \hat{T}_{Hz}(\Omega_{kP})$$

Noise sample covariance sample mean

$$\hat{C}_Z(kP) = (1 + \delta^2)\hat{C}_V(kP)$$

where typically $\delta = 0.5$ giving a 1 dB increase in uncertainty
4. Improved nonparametric models for LTI systems

4.f Periodic excitations — steady state (cont’d)

Robust method – linear plant – \( r(t) \) not needed

- as many experiments as inputs with orthogonal multisines \((P \geq 2 \text{ periods})\)
  - transient removal via LPM at non-excited lines
  - result: sample means + noise covariance estimate at excited lines

- FRF estimate \( \hat{G}(\Omega_k) = \hat{Y}(k)\hat{U}^{-1}(k) + \text{noise covariance} \)

Robust method – nonlinear plant – \( r(t) \) needed

- as many experiments as inputs with full random orthogonal multisines \((P \geq 2 \text{ periods})\)
  - transient removal via LPM at non-excited lines
  - result: sample means + noise covariance estimate at excited lines

- repeat the experiments for \( M \geq 2 \) independent random phase realisations

  - project input-output DFT spectra on reference DFT spectra
  - sample mean and total sample covariance (noise + NL distortions)

  - FRF estimate \( \hat{G}(\Omega_k) = \hat{Y}_R(k)\hat{U}_R^{-1}(k) + \text{noise and total covariances} \)
4. Improved nonparametric models for LTI systems

4.f Periodic excitations — steady state (cont’d)

Fast method – (non)linear plant – \( r(t) \) needed

- one experiment with random phase multisines \( (P \geq 2 \) periods)
  - transient removal via LPM at non-excited lines
  - result: sample means + noise covariance estimate at excited lines

- apply LPM to the sample means at the excited frequencies
  - local polynomial approximation FRF (note: no transient term!)

\[
\hat{Z}(kP) = \begin{bmatrix} G_{ry}(\Omega_{kP}) \\ G_{ru}(\Omega_{kP}) \end{bmatrix} \hat{R}(k) + \hat{V}_Z(kP) \left( \begin{bmatrix} \hat{Y}_S(k) \\ \hat{U}_S(k) \end{bmatrix} \right)
\]

- FRF estimate \( \hat{G}(\Omega_{kP}) = \hat{G}_{ry}(\Omega_{kP})\hat{G}_{ru}^{-1}(\Omega_{kP}) \) + noise and total covariances

Note: SISO LTI plant \( \Rightarrow \hat{G}(\Omega_{kP}) = \hat{Y}(kP)/\hat{U}(kP) \)
### 4. Improved nonparametric models for LTI systems

4.f Periodic excitations — steady state (cont’d)

Comparison robust and fast methods

<table>
<thead>
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<th>Fast</th>
<th>Robust</th>
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<td>transient + FRF</td>
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<tr>
<td>Measurements</td>
<td>1 experiment $P \geq 2$ periods</td>
<td>$n_u$ experiments $P \geq 2$ periods</td>
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<td>Frequency resolution</td>
<td>$\frac{1}{P \times T}$</td>
<td>$\frac{1}{n_u \times P \times M \times T}$</td>
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$(T = \text{total measurement time})$

Note: factor $P$ loss in frequency resolution of the fast method w.r.t. arbitrary excitations
4. Improved nonparametric models for LTI systems

4.g Measurement example: longitudinal vibrations of a plexiglass beam

\[ u(t) \rightarrow \begin{array}{c} y(t, x) \\ \text{plexiglass beam} \\ 1 \text{ cm} \times 2 \text{ cm} \times 1.983 \text{ m} \end{array} \]

Force-to-acceleration \( M = 25, P = 10 \)

\[ \frac{\text{Force-to-acceleration}}{M} = \frac{1}{P} = \frac{1}{10} \]

\[ \begin{align*}
\text{measured FRF} &
\text{total variance} \\
\text{noise variance} &
\end{align*} \]
4. Improved nonparametric models for LTI systems

4.h Periodic excitations — transient regime

Note: \( \text{T}_{G_y} \) and \( \text{T}_{G_U} \) cause a \textit{bias} and increase the \textit{variability} of the FRF measurement.

Conclusion: The robust and fast methods for periodic excitations are robust to plant transient (leakage) errors.
4. Improved nonparametric models for LTI systems

4.i Measurement example: flexural vibrations of a steel beam

Experiment (see also p. 177)

- 1 experiment with a random phase multisine excitation in the band [0 Hz, 6 kHz]
- $P = 50$ consecutive periods

Processing

1. subtract the last period from the other signal periods
2. calculate the rms value of the residuals over each period

![Graph showing rms residual (dB) vs. period number]
4. Improved nonparametric models for LTI systems

4.i Measurement example: flexural vibrations of a steel beam (cont’d)

![Graphs showing FRFs and variances for different periods and models.](image)
4. Improved nonparametric models for LTI systems

4.j Summary local polynomial methods

Arbitrary excitations

- full frequency resolution (1 experiment)
- no distinction between noise and nonlinear distortions
- local polynomial approximation of the FRF and the transient (leakage) errors

Periodic excitations - fast method

- 1/2 full frequency resolution (1 experiment, 2 periods)
- separation between noise and nonlinear distortions
- local polynomial approximation of the FRF and the transient (leakage) errors

Periodic excitations - robust method

- 1/(4n_u) full frequency resolution (n_u experiments, 2 periods, 2 realisations)
- separation between noise and nonlinear distortions
- local polynomial approximation of the transient (leakage) errors only
4. Improved nonparametric models for LTI systems

4.k Time-variant systems – arbitrary excitations

Goal

Variance reduction FRF measurement by modelling the time-variation $Y_{TV}(k)$ via a series expansion of the time-variant FRF w.r.t. time

$$G(j\omega, t) = \sum_{p=0}^{N_b} G_p(j\omega)f_p(t) \text{ for } t \in [0, T]$$

with $f_p(t)$ Legendre polynomials of order $p$
4. Improved nonparametric models for LTI systems

4.k Time-variant systems – arbitrary excitations (cont’d)

\[ G(j\omega, t) = G_0(j\omega) + \sum_{p=1}^{N_b} G_p(j\omega)f_p(t) \]

\[ y(t) = L^{-1}\{ G(s, t)U(s) \} = L^{-1}\{ G_0(s)U(s) \} + \sum_{p=1}^{N_b} L^{-1}\{ G_p(s)U(s) \}f_p(t) \]
4. Improved nonparametric models for LTI systems

4.k Time-variant systems – arbitrary excitations (cont’d)

Basic idea modelling time-variation

\[
\begin{align*}
\begin{bmatrix}
u(t) \\
u_1(t) \\
\vdots \\
u_{Nb}(t)
\end{bmatrix} & \rightarrow \\
\begin{bmatrix}
y(t) \\
y_1(t) \\
\vdots \\
y_{Nb}(t)
\end{bmatrix}
\end{align*}
\]

SISO LTV \quad \text{MISO LTI}
4. Improved nonparametric models for LTI systems

4.k Time-variant systems – arbitrary excitations (cont’d)

Solve the MISO LTI problem using the LPM, and transform back to the SISO LTV problem

\[
G_{\text{BLTI}}(s) = G_0(s) = H_0(s) + \frac{2}{T} \sum_{p=0}^{\lfloor (N_b-1)/2 \rfloor} H_{2p+1}^{(1)}(s) + \frac{4}{T^2} \sum_{p=1}^{\lfloor N_b/2 \rfloor} \beta_{2p} H_{2p}^{(2)}(s) + O(T^{-3})
\]

\[
Y_{\text{TV}}(k) = Y(k) - (G_0(j\omega_k)U(k) + T_{G_0}(j\omega_k)) = Y(k) - G_0(j\omega_k)U(k) + O(T^{-1/2})
\]

with \( \beta_{2p} = 1.5 + 2.5(p-1) + (p-1)^2 \)

Note
- numerical differentiation \( H_p(j\omega) \)
- \( T_{G_0}(j\omega_k) = O(T^{-1/2}) \) is unknown, but zero for periodic \( u(t) \) under steady state
4. Improved nonparametric models for LTI systems

4.1 Time-variant systems – periodic excitations

Extension to a class of nonlinear time-variant systems

Output NLTV split in

- nonlinear time-invariant part
- linear time-variant part

Approximation reasonable for weakly nonlinear, weakly time-variant systems
4. Improved nonparametric models for LTI systems

4.1 Time-variant systems – periodic excitations (cont’d)

\[ G_0(s) \]

with \( G_0(s) \) the BLA of the NL PISPO system
4. Improved nonparametric models for LTI systems

4.1 Time-variant systems – periodic excitations (cont’d)

Properties

- $Y_{TV}(k)$ is uncorrelated with – but not independent of – $U(k)$
- $Y_{S}(k)$ is uncorrelated with – but not independent of – $U(k)$
- $Y_{S}(k)$ is uncorrelated with – but not independent of – $Y_{TV}(k)$
4. Improved nonparametric models for LTI systems

4.1 Time-variant systems – periodic excitations (cont’d)

Noisy output observations

\[ u(t) \xrightarrow{\text{NLTV}} y(t) + n_y(t) = y_{TV}(t) + y_s(t) + n_y(t) \]

Observation: \( Y_{TV}(k), Y_s(k) \) and \( N_Y(k) \) have similar first and second order properties

How to distinguish \( y_{TV}(t), y_s(t) \) and \( n_y(t) \)?

Key properties

- \( y_s(t) \) has the same periodicity as \( u(t) \) and is nonlinearly related to \( u(t) \)
- \( y_{TV}(t) \) is linearly related to \( u(t) \)
- \( n_y(t) \) is independent of \( u(t) \)
4. Improved nonparametric models for LTI systems

4.1 Time-variant systems – periodic excitations (cont’d)

Noisy output observations

\[
\begin{align*}
& n_y(t) \\
\rightarrow u(t) \quad \text{NLTV} \quad \rightarrow y(t) \\
\end{align*}
\]

\[
\begin{align*}
 & y(t) \\
\rightarrow G_{\text{BLT1}}(s) \quad \rightarrow y_{\text{BLT1}}(t) \\
\end{align*}
\]

DFT of \( P = 2 \) consecutive periods

\[
\begin{align*}
& y_{TV}(t) + y_s(t) + n_y(t) \\
\rightarrow Y_{\text{BLT1}}(k) \quad Y_{TV}(k) \quad Y_s(k) \quad N_y(k) \\
\end{align*}
\]
4. Improved nonparametric models for LTI systems

4.1 Time-variant systems – periodic excitations (cont’d)

\[ y(t) = y_s(t) + n_y(t) \]

\[ H_0(s) \]

\[ u(t) \]

\[ f_1(t) \]

\[ f_{N_b}(t) \]

\[ u_1(t) \]

\[ u_{N_b}(t) \]

\[ H_1(s) \]

\[ H_{N_b}(s) \]

\[ Y(k) \]

\[ U(k) \]

\[ Y_s(k) \]

\[ H_0(j\omega_k)U(k) \]

\[ \sum_{p=1}^{N_b} H_p(j\omega_k)U_p(k) + T_H(j\omega_k) \]

\[ N_Y(k) \]

\[ Y_S(k) \]
4. Improved nonparametric models for LTI systems

4.1 Time-variant systems – periodic excitations (cont’d)

3-step procedure

1. non-excited DFT lines \( k \neq rP \)
   - estimation \( H_p(j\omega_k) \), \( p = 1, 2, \ldots, N_b \), and \( \text{var}(N_Y(k)) \) using LPM

2. excited DFT lines \( k = rP \)
   - removal time-variant effects: \( Y(k) - \sum_{p=1}^{N_b} H_p(j\omega_k)U_p(k) - T_H(j\omega_k) \)
   - estimation \( H_0(j\omega_k) \) and \( \text{var}(N_Y(k) + Y_S(k)) \) using LPM

3. excited DFT lines \( k = rP \)
   - calculation \( G_{BLTI}(j\omega_k) \) and \( Y_{TV}(k) \) from \( H_p(j\omega_k) \), \( p = 0, 1, \ldots, N_b \)
4. Improved nonparametric models for LTI systems

4.m Measurement example: time-variant electronic circuit operating in feedback
4. Improved nonparametric models for LTI systems

4.m Measurement example: time-variant electronic circuit operating in feedback (cont’d)

Reference signal \( r(t) \): random phase multisine excitation

- 522 frequencies in the band [229 Hz, 40 kHz] with an rms value of 100 mV
- \( T = PN/f_s \) with \( P = 2 \), \( N = 8192 \) and \( f_s = 625 \text{ kHz} \)

Variation scheduling parameter \( p(t) \) over \( P = 2 \) periods of \( r(t) \): 1 LTV and 3 LTI experiments
4. Improved nonparametric models for LTI systems

4.m Measurement example: time-variant electronic circuit operating in feedback (cont’d)

\[ H_{ru,1}(j\omega_k) \]

\[ H_{ru,2}(j\omega_k) \]

\[ H_{ry,1}(j\omega_k) \]

\[ H_{ry,2}(j\omega_k) \]

- dynamics input-output
- time-variant branches
- noise variance
4. Improved nonparametric models for LTI systems

4.m Measurement example: time-variant electronic circuit operating in feedback (cont’d)

Note: the feedback loop suppresses (i) the nonlinear distortions, and (ii) the time-variation

\[ Y_S = \tilde{Y}_S - G_{\text{BLTI}} \tilde{U}_S \]
\[ Y_{TV} = \tilde{Y}_{TV} - G_{\text{BLTI}} \tilde{U}_{TV} \]
4. Improved nonparametric models for LTI systems

4.m Measurement example: time-variant electronic circuit operating in feedback (cont’d)

![Graph showing the amplitude in dB over frequency for different variances](image)

- $G_{BLTI}(j\omega_k)$
- Noise variance
- Total variance
4. Improved nonparametric models for LTI systems

4.m Measurement example: time-variant electronic circuit operating in feedback (cont’d)
4. Improved nonparametric models for LTI systems

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5. Estimation parametric plant model with known noise model

5.a Frequency domain data

Noisy input/output observations

\[
\begin{align*}
Y(k) &= Y_0(k) + N_Y(k) \\
U(k) &= U_0(k) + N_U(k)
\end{align*}
\]

with known

\[
\begin{align*}
\sigma_Y^2(k) &= \text{var}(N_Y(k)) = \mathbb{E}\{|N_Y(k)|^2\} \\
\sigma_U^2(k) &= \text{var}(N_U(k)) = \mathbb{E}\{|N_U(k)|^2\} \\
\sigma_{YU}(k) &= \text{covar}(N_Y(k), N_U(k)) = \mathbb{E}\{N_Y(k)N_U(k)\}
\end{align*}
\]

- periodic signals
  \[U_0(k) = U_g(k)\]

- arbitrary signals
  \[U_0(k) = U_g(k) + N_g(k)\]
5. Estimation parametric plant model with known noise model

5.a Frequency domain data (cont’d)

Note:

FRF measurements
\[ G(j\omega_k) = G_0(j\omega_k) + N_G(k) \]

with known \( \sigma^2_G(k) = \text{var}(N_G(k)) = \mathbb{E}\{|N_G(k)|^2\} \)

are a special case of noisy input/output observations
\[ Y(k) = G(k), \ U(k) = 1, \ N_Y(k) = N_G(k), \ \text{and} \ N_U(k) = 0 \]
5. Estimation parametric plant model with known noise model

5.b Maximum likelihood estimator

Define

\[ Z(k) = [Y(k), U(k)]^T \text{ and } N_Z(k) = [N_Y(k) \ N_U(k)]^T \]

Assumption

\( N_Z(k) \) has zero mean and is independent (over \( k \)), circular complex normally distributed

Gaussian likelihood function

\[
\begin{align*}
f_Z(Z|\theta, Z_p) & = \prod_{k=1}^{F} f_{Z(k)}(Z(k)|\theta, Z_p(k)) \\
& = \frac{1}{\pi^{2F} \prod_{k=1}^{F} \det(C_Z(k))} \exp\left(-\sum_{k=1}^{F} (Z(k) - Z_p(k))^H C_Z^+(k)(Z(k) - Z_p(k)) \right)
\end{align*}
\]

with

\[
Z_p(k) = \begin{bmatrix} Y_p(k) \\ U_p(k) \end{bmatrix} \text{ and } C_Z(k) = \text{Cov}(N_Z(k)) = \begin{bmatrix} \sigma_Y^2(k) & \sigma_{YU}(k) \\ \sigma_{UY}(k) & \sigma_U^2(k) \end{bmatrix}
\]

and where \( C_Z(k) \) can be singular (e.g. feedback with process noise and no measurement noise)

Note:

- \( C_Z^+(k) \) is the Moore-Penrose pseudo-inverse of \( C_Z(k) \)
5. Estimation parametric plant model with known noise model

5.b Maximum likelihood estimator (cont’d)

• Periodic signals

Gaussian ML cost function

\[
V_{\text{ML}}(\theta, Z_p, Z) = \sum_{k=1}^{F} (Z(k) - Z_p(k))^H C_Z^\pm (Z(k) - Z_p(k)) + \sum_{k=1}^{F} \lambda_k [1, -G(\Omega_k, \theta)] Z_p(k)
\]

where \( G(\Omega, \theta) \) is parametrized according to one of the plant models in chapter 3

Eliminate \( Z_p = [Z_p(1), Z_p(2), \ldots, Z_p(F)]^T \), \( \lambda \Rightarrow \)

\[
V_{\text{ML}}(\theta, Z) = \sum_{k=1}^{F} \frac{|e(\Omega_k, \theta, Z(k))|^2}{\sigma_e^2(\Omega_k, \theta)}
\]

where

\[
e(\Omega_k, \theta, Z(k)) = Y(k) - G(\Omega_k, \theta) U(k)
\]
\[
\sigma_e^2(\Omega_k, \theta) = \sigma_Y^2(k) + |G(\Omega_k, \theta)|^2 \sigma_U^2(k) - 2 \text{Re}(\overline{G(\Omega_k, \theta)} \sigma_Y^2 U(k))
\]

• Arbitrary signals

Same cost function with

\[
e(\Omega_k, \theta, Z(k)) = Y(k) - G(\Omega_k, \theta) U(k) - T_G(\Omega_k, \theta)
\]
5. Estimation parametric plant model with known noise model

5.b Maximum likelihood estimator (cont’d)

• Discussion

\[ V_{\text{ML}}(\theta, Z) = \sum_{k=1}^{F} \frac{|e(\Omega_k, \theta, Z(k))|^2}{\sigma_e^2(\Omega_k, \theta)} \]

with

\[
\begin{align*}
e(\Omega_k, \theta, Z(k)) &= Y(k) - G(\Omega_k, \theta)U(k) - T_G(\Omega_k, \theta) \\
\sigma^2_e(\Omega_k, \theta) &= \sigma^2_Y(k) + |G(\Omega_k, \theta)|^2 \sigma^2_U(k) - 2\text{Re}(\bar{G}(\Omega_k, \theta) \sigma^2_{YU}(k))
\end{align*}
\]

- suppresses frequency bands with poor signal-to-noise ratio

- relative importance input noise w.r.t. output noise (uncorrelated case \( \sigma^2_{YU} = 0 \))

\[
\frac{|G(\Omega_k, \theta)|^2 \sigma^2_Y(k)}{\sigma^2_Y(k)}
\]

- significance correlation between input/output errors

\[
\rho(k) = \frac{-2\text{Re}(\sigma^2_{YU}(k)\bar{G}(\Omega_k, \theta))}{\sigma^2_Y(k) + |G(\Omega_k, \theta)|^2 \sigma^2_U(k)} \quad \text{with} \quad \begin{cases} 
\rho(k) < 0 \Rightarrow \text{decrease } \sigma^2_e(\Omega_k, \theta) \\
\rho(k) > 0 \Rightarrow \text{increase } \sigma^2_e(\Omega_k, \theta)
\end{cases}
\]
5. Estimation parametric plant model with known noise model

5.2 Maximum likelihood estimator (cont’d)

Discussion (cont’d)

- periodic signals + open loop + generator noise dominant in a certain frequency band
  \[ \sigma_e^2(\Omega_k, \theta) \approx |G_0(\Omega_k) - G(\Omega_k, \theta)|^2 \sigma_g^2 \Rightarrow \text{high weighting in cost function} \]
  \[ \Rightarrow \text{generator noise does not increase the variability of the estimates} \]

- arbitrary signals: generator noise is a part of the excitation

- neglect transients

  \[ V_{\text{ML}}(\theta, Z) = \sum_{k=1}^{F} \frac{|e(\Omega_k, \theta, Z(k))|^2}{\sigma_e^2(\Omega_k, \theta)} = \sum_{k=1}^{F} \frac{|G(\Omega_k) - G(\Omega_k, \theta)|^2}{\sigma_e^2(\Omega_k, \theta)} |U(k)|^2 \]
  \[ \Rightarrow \text{estimate depends on power spectrum input only} \]
  \[ \Rightarrow \text{same results for multisine or pulse excitation} \]
  note: in practise the peak value of the excitation is limited (nonlinearities!)
5. Estimation parametric plant model with known noise model

5.b Maximum likelihood estimator (cont’d)

- Properties

  - Consistency

\[
V_F(\theta) = \mathbb{E}\left\{ \frac{1}{F} V_{ML}(\theta, Z) \right\} = \mathbb{E}\left\{ \frac{1}{F} \sum_{k=1}^{F} \frac{|e(\Omega_k, \theta, Z_0(k))|^2}{\sigma_e^2(\Omega_k, \theta)} \right\} + 1
\]

expected value cost function \(V_F(\theta)\) is minimal in \(\theta = \theta_0 \Rightarrow\) consistent

- Convergence rate

\[
\hat{\theta}(Z) = \theta_0 + \delta_\theta(Z) + b_\theta(Z)
\]

\[
\delta_\theta(Z) = -V_F^{-1}(\theta_0)V_F^T(\theta_0, Z)
\]

with

\[
\delta_\theta(Z) = O_p(F^{-1/2}) \text{ and } \mathbb{E}\{ \delta_\theta(Z) \} = 0: \text{dominating stochastic error}
\]

\[
b_\theta(Z) = O_p(F^{-1}): \text{bias error}
\]

- Asymptotic normality

\[
\sqrt{F}(\hat{\theta}(Z) - \theta_0) \in \text{As}N(0, \text{Cov}(\sqrt{F}\delta_\theta(Z)))
\]

with

\[
\text{Cov}(\sqrt{F}\delta_\theta(Z)) = V_F^{-1}(\theta_0)Q_F(\theta_0)V_F^{-1}(\theta_0)
\]

\[
Q_F(\theta_0) = F\mathbb{E}\{ V_F^T(\theta_0, Z)V_F'(\theta_0, Z) \}
\]
5. Estimation parametric plant model with known noise model

5.1 Maximum likelihood estimator (cont’d)

- Properties (cont’d)
  - Cramér-Rao lower bound
    \[
    \text{Cov}(\sqrt{F\hat{\theta}}) \geq V_F''(\theta_0)
    \]
    \[
    \Rightarrow \text{the ML estimator is inefficient; inefficiency term is, however, small (Pintelon and Mei, 2007)}
    \]
  - Special cases: input noise only, output noise only, totally correlated input/output errors
    \[
    Q_F(\theta_0) = V_F''(\theta_0)
    \]
    \[
    \Rightarrow \text{the ML estimator is asymptotically efficient}
    \]
  - Robustness: mixing non-Gaussian noise \( \Rightarrow \) consistent, convergence rate, asymptotic normality remain valid
  - Rational transfer function parametrization: estimate invariants independent of the parameter constraint chosen
    \[
    G(\Omega, \lambda \theta) = G(\Omega, \theta) \Rightarrow V_{ML}(\lambda \theta, Z) = V_{ML}(\theta, Z)
    \]
  - Model errors (unmodelled dynamics, nonlinear distortions): replace everywhere \( N_Z \rightarrow \lambda N_Z \)
    \[
    \theta_0 \rightarrow \tilde{\theta}(Z_0) = \arg \min_{\theta} \mathbb{E} \{ V_{ML}(\theta, Z) \} \text{ with } \tilde{\theta}(Z_0) \text{ independent of the noise level } \lambda
5. Estimation parametric plant model with known noise model

5.c Calculation ML estimates

- Newton-Gauss

Basic algorithm

\[ \text{Re}(J^{(i)H}J^{(i)}) \Delta \theta^{(i+1)} = -\text{Re}(J^{(i)H} \varepsilon^{(i)}) \]

where

\[ \varepsilon^{(i)} = \frac{e(\Omega_k, \theta^{(i)}, Z(k))}{\sigma_e(\Omega_k, \theta^{(i)})} \quad \text{and} \quad J^{(i)}_{[k,r]} = \frac{\partial \varepsilon_{[k]}^{(i)}}{\partial \theta_{[r]}^{(i)}} \]

Numerical stable implementation

\[ J^{(i)}_{\text{re}} \Delta \theta^{(i+1)} = -\varepsilon_{\text{re}}^{(i)} \Rightarrow \Delta \theta^{(i+1)} = -(J^{(i)}_{\text{re}})^+ \varepsilon_{\text{re}}^{(i)} = -V^{(i)}(\Sigma^{(i)}) + U^{(i)T} \varepsilon_{\text{re}}^{(i)} \]

where

\[ X_{\text{re}} = \begin{bmatrix} \text{Re}(X) \\ \text{Im}(X) \end{bmatrix} \quad \text{and} \quad J^{(i)}_{\text{re}} = U^{(i)} \Sigma^{(i)} V^{(i)T} \]

Notes:

- continuous-time, scale angular frequencies by median frequency set
  \[ \omega_{\text{med}} = \text{med}\{\omega_1, \omega_2, \ldots, \omega_F\} \Rightarrow b_r s^r = (b_r \omega_{\text{med}}^r) (s / \omega_{\text{med}})^r \]
- leave all parameters free and impose in each iteration step the parameter constraint, for example, \[ \|\theta\|_2^2 = 1 \]
- normalize the columns of \[ J^{(i)}_{\text{re}} \] by their 2-norm
5. Estimation parametric plant model with known noise model

5.c Calculation ML estimates (cont’d)

• Levenberg-Marquardt

Basic algorithm

\[(J_{re}^{(i)T}J_{re}^{(i)} + \lambda^2 I)\Delta \theta^{(i+1)} = -J_{re}^{(i)T} \varepsilon_{re}^{(i)}\]

with

\[\lambda_{\text{start}} = \sigma_{\text{max}}(J_{re}) / 100\]

cost function decreases: \(\lambda \rightarrow 0.4\lambda\)

cost function increases: \(\lambda \rightarrow 10\lambda\)

Numerical stable implementation for the constraint \(\|\theta\|_2^2 = 1\)

\[\Delta \theta^{(i+1)} = -V^{(i)} A^{(i)} U^{(i)T} \varepsilon_{re}^{(i)}\]

with

\[J_{re} = U \text{diag}({\sigma_1, \sigma_2, \ldots, \sigma_{n_\theta-1}, 0}) V^T\]

\[A = \text{diag} \left( \frac{\sigma_1}{\sigma_1^2 + \lambda^2}, \frac{\sigma_2}{\sigma_2^2 + \lambda^2}, \ldots, \frac{\sigma_{n_\theta-1}}{\sigma_{n_\theta-1}^2 + \lambda^2}, 0 \right)\]
5. Estimation parametric plant model with known noise model

5.6 Calculation covariance matrix ML estimates

Fix a coefficient, for example, of the largest denominator power of $\Omega$, and remove the corresponding column in the Jacobian matrix

$$\text{Cov}(\hat{\theta}) \approx (2\text{Re}(J^H J))^{-1} = (2J_{re}^T J_{re})^{-1} = \frac{1}{2}(V\Sigma^{-1})(V\Sigma^{-1})^T$$

with

$$J_{[k,r]} = \frac{\partial \varepsilon_{[k]}}{\partial \hat{\theta}_{[r]}}$$

$$\varepsilon_{[k]} = \frac{e(\Omega_k, \theta, Z(k))}{\sigma_e(\Omega_k, \theta)}$$

$$J_{re} = \begin{bmatrix} \text{Re}(J) \\ \text{Im}(J) \end{bmatrix} = U\Sigma V^T$$

Note: perform the calculations on the frequency normalized parameters for $s$- and $\sqrt{s}$-domains.
5. Estimation parametric plant model with known noise model

5.e Starting values

• Basic idea for any parametrization

  Two possibilities
  - rational transfer function + linear least squares + back transformation
  - state space + subspace algorithm + back transformation

• Linear least squares

\[
V_{\text{LS}}(\theta, Z) = \sum_{k=1}^{F} |e(\Omega_k, \theta, Z(k))|^2 = \sum_{k=1}^{F} |A(\Omega_k, \theta)Y(k) - B(\Omega_k, \theta)U(k) - I_G(\Omega_k, \theta)|^2
\]

Properties

- inconsistent

\[
\mathbb{E}\{ V_{\text{LS}}(\theta, Z) / F \} = \frac{1}{F} \sum_{k=1}^{F} |e(\Omega_k, \theta, Z_0(k))|^2 + \frac{1}{F} \sum_{k=1}^{F} \sigma^2_e(\Omega_k, \theta)
\]

  is not minimal in \( \theta_0 \)

- overemphasizes high frequency errors

- estimate is constraint dependent since \( V_{\text{LS}}(\lambda \theta, Z) = \lambda^2 V_{\text{LS}}(\theta, Z) \)

- a well chosen weighting can be added \( \Rightarrow \) WLS

- \( \tilde{\theta}(Z_0) = \text{arg min} \{ V_{\text{LS}}(\theta, Z) \} \) depends on the noise level
5. Estimation parametric plant model with known noise model

5.e Starting values (cont’d)

- Iterative quadratic maximum likelihood (IQML)

  Iterative algorithm

  \[ V_{\text{IQML}}(\theta^{(i+1)}, Z) = \sum_{k=1}^{F} \frac{|e(\Omega_k, \theta^{(i+1)}, Z(k))|^2}{\sigma_e^2(\Omega_k, \theta^{(i)})} \]

  started from (W)LS solution

- Other methods
  - generalized total least squares (consistent + amplification high frequency errors)
  - bootstrapped total least squares (consistent + close to ML + not self starting)
  - subspace (consistent, frequency weighting not clear)
5. Estimation parametric plant model with known noise model

5.f Simulation example: 2nd order continuous-time system

\[ G(s, \theta_0) = \frac{1}{1 + s + s^2} \]

Simulation parameters
- \( U_0(k) = 1 \)
- \( N_U(k), N_Y(k) \in N^c(0, 0.02) \)
- \( F = 100 \) in the band \([0.1, 10]/(2\pi)^2\) Hz
5. Estimation parametric plant model with known noise model

5.f Simulation example: 2nd order continuous-time system (cont’d)

Observe the influence of the constraint on the LS estimate:

- $a_0 = 1 \Rightarrow$ underbiased
- $b_0 = 1 \Rightarrow$ overbiased
5. Estimation parametric plant model with known noise model

5.f Simulation example: 2nd order continuous-time system (cont’d)

Nonlinear least squares

NLS-I/O:

\[ \sum_{k=1}^{F} |Y(k) - G(\Omega_k, \theta)U(k)|^2 \]

NLS-FRF:

\[ \sum_{k=1}^{F} \left| \frac{Y(k)}{U(k)} - G(\Omega_k, \theta) \right|^2 \]

\[ f \ (\text{Hz}) \]

\[ \text{Amplitude (dB)} \]

\[ \text{Amplitude (dB) error} \]
5. Estimation parametric plant model with known noise model

5.f Simulation example: 2nd order continuous-time system (cont’d)

Notes:
- LS uses no noise information
- IQML, and ML use noise information
5. Estimation parametric plant model with known noise model

5.g Measurement example: q-axis impedance 3.4 MW synchronous machine

\[ n_b = 4, n_a = 3 \]

\[ b_0 = 1 \]
5. Estimation parametric plant model with known noise model

5.h Measurement example: flight flutter data

\[ n_b = 11, n_a = 10 \]
5. Estimation parametric plant model with known noise model

5.i Model validation

• Comparison with measured FRF

Compare \( G(\Omega_k, \hat{\theta}) \) to the nonparametric LPM estimate (see Chapter 4)

\[
\begin{align*}
\hat{G}_{\text{fast}}(\Omega_k), \hat{G}_{\text{robust}}(\Omega_k) & \quad \text{for periodic signals} \\
\hat{G}_{\text{arb}}(\Omega_k) & \quad \text{for arbitrary signals}
\end{align*}
\]

taking into account the uncertainty \( \sigma^2_G(k) \) of the LPM estimate

• Analysis cost function

No modelling errors

\[
\begin{align*}
\mathbb{E} \{ V_{\text{ML}}(\hat{\theta}_{\text{ML}}(Z), Z) \} & \approx V_{\text{noise}} \\
\text{var}(V_{\text{ML}}(\hat{\theta}_{\text{ML}}(Z), Z)) & \approx V_{\text{noise}}
\end{align*}
\]

with \( V_{\text{noise}} = F - n_\theta / 2 \)

\( n_\theta = \text{number of free parameters} \)

Test

\[
\begin{align*}
V_{\text{ML}}(\hat{\theta}_{\text{ML}}(Z), Z) > V_{\text{noise}} + 2 \sqrt{V_{\text{noise}}} & \Rightarrow \text{modelling errors} \\
V_{\text{ML}}(\hat{\theta}_{\text{ML}}(Z), Z) \in [V_{\text{noise}} - 2 \sqrt{V_{\text{noise}}}, V_{\text{noise}} + 2 \sqrt{V_{\text{noise}}} ] & \Rightarrow \text{no modelling errors} \\
V_{\text{ML}}(\hat{\theta}_{\text{ML}}(Z), Z) < V_{\text{noise}} - 2 \sqrt{V_{\text{noise}}} & \Rightarrow \text{wrong noise variances?}
\end{align*}
\]
5. Estimation parametric plant model with known noise model

5.i Model validation (cont’d)

- Whiteness test residuals measured minus modelled FRF

Sample correlation residuals

\[
\hat{R}_{\delta\delta}(m) = \frac{1}{F - m} \sum_{k=1}^{F-m} \delta(\Omega_k, \hat{\theta}) \bar{\delta}(\Omega_{k+m}, \hat{\theta})
\]

with \( \delta(\Omega_k, \hat{\theta}) = \frac{\hat{G}(\Omega_k) - G(\Omega_k, \hat{\theta})}{\sigma_G(\Omega_k)} \)

No unmodelled dynamics

- 95% confidence bound \( \sqrt{3} \text{std}(\hat{R}_{\delta\delta}(m)) = \sqrt{3/(F - m)} \)

- \( \hat{R}_{\delta\delta}(m) \) is asymptotically \( (F \to \infty) \) a Dirac function

\[
\begin{align*}
\hat{R}_{\delta\delta}(m) &= O(F^{-1/2}) \quad \text{for} \ m \neq 0 \\
\hat{R}_{\delta\delta}(0) &\geq 1
\end{align*}
\]

- \( \hat{R}_{\delta\delta}(0) = 1 \) and noise (co-)variances \( \Rightarrow \) no nonlinear distortions

- \( \hat{R}_{\delta\delta}(0) > 1 \) and noise (co-)variances \( \Rightarrow \) nonlinear distortions

- (co-)variances noise + stochastic nonlinear distortions \( \Rightarrow \hat{R}_{\delta\delta}(0) = 1 \)
5. Estimation parametric plant model with known noise model

5.j Measurement example: flexible robot arm

- meas. FRF
- estim. model
- std FRF
- residual meas. - model

\[
\begin{align*}
V_{ML} &= 4964.8 \\
V_{\text{noise}} &= 95.5 \\
V_{ML} &= 220.5 \\
V_{\text{noise}} &= 93.5
\end{align*}
\]

\[ F = 100 \]

Observation: \( \hat{R}_{\delta\delta}(0) > 1 \) and \( \sigma_G^2 = \text{var}(N_G) \Rightarrow \) nonlinear distortions
5. Estimation parametric plant model with known noise model

5.k Multivariable systems

- ML cost function

\[
V_{ML}(\theta, Z) = \sum_{k=1}^{F} e^H(\Omega_k, \theta, Z(k)) C_e^{-1}(\Omega_k, \theta) e(\Omega_k, \theta, Z(k))
\]

with

\[
\begin{align*}
\begin{cases}
e(\Omega_k, \theta, Z(k)) = Y(k) - G(\Omega_k, \theta) U(k) - T_G(\Omega_k, \theta) \\
C_e(\Omega_k, \theta) = C_Y(k) + G(\Omega_k, \theta) C_U(k) G^H(\Omega_k, \theta) - 2 \text{herm}(C_{YU}(k) G^H(\Omega_k, \theta))
\end{cases}
\end{align*}
\]

and \( \text{herm}(X) = (X + X^H)/2 \)

- Difficulty: calculation Jacobian matrix

\[
J_{[k, r]} = \frac{\partial C_e^{-1/2}(\Omega_k, \theta) e(\Omega_k, \theta, Z(k))}{\partial \theta_{[r]}}
\]

- Solution: pseudo-Jacobian (see Guillaume and Pintelon, 1996)

\[
J_{+[k, r]} = C_e^{-1/2}(\Omega_k, \theta) \frac{\partial e(\Omega_k, \theta, Z(k))}{\partial \theta_{[r]}} + \ldots
\]

\[
-\frac{1}{2} C_e^{-1/2}(\Omega_k, \theta) \frac{\partial C_e(\Omega_k, \theta)}{\partial \theta_{[r]}} C_e^{-1}(\Omega_k, \theta) e(\Omega_k, \theta, Z(k))
\]
5. Estimation parametric plant model with known noise model

5.1 Model selection

Problem statement

![Graph showing amplitude vs frequency for true 5th order FIR system, estimated 5th order FIR model, and estimated 50th order FIR model.](image-url)
5. Estimation parametric plant model with known noise model

5.1 Model selection (cont’d)

Problem statement

Increasing model complexity: cost function decreases AND model variability increases

“Optimal” model complexity: balance between model errors AND model variability

Ideal situation

Identification and validation data sets are available

Identification data set only

Model selection criteria

\[
\left( \frac{2F}{2F - n_\theta} \right) V_F(\hat{\theta}(Z), Z)(1 + p(n_\theta, F))
\]

with \( V_F(\hat{\theta}(Z), Z) \) the ML cost function and \( p(n_\theta, F) \) the penalty

\[
p(n_\theta, F) = \begin{cases} 
\frac{2n_\theta}{2F - n_\theta} & \text{AIC} \\
\log (2(n_y + n_u)F)n_\theta & \text{MDL (EIV)} \\
\frac{\log (2n_yF)n_\theta}{2F - n_\theta} & \text{MDL (OE)}
\end{cases}
\]
5. Estimation parametric plant model with known noise model

5.m Simulation example: SISO-FIR system

\[ G(z^{-1}, \theta) = \sum_{r=0}^{R} g_r z^{-r} \text{ with } \theta = [g_0, g_1, \ldots, g_R]^T \]

Input/output DFT spectra

\[ U_0(k) = 1, Y_0(k) = G_0(z_k^{-1}), \sigma_Y(k) = \sigma_U(k) = 0.05, \sigma_{YU}(k) = 0, F = 40 \]

Results 1000 runs

<table>
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<th>model order ( R )</th>
<th>AIC</th>
<th>MDL</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
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</tr>
<tr>
<td>mean value ± std</td>
<td>19.3 ± 0.79</td>
<td>19.1 ± 0.51</td>
</tr>
</tbody>
</table>
5. Estimation parametric plant model with known noise model

5.n References
6. Estimation parametric plant model with estimated nonparametric noise model

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6. Estimation parametric plant model with estimated nonparametric noise model

6.a Classical approach: basic idea

- Extract noise covariance matrix from the experimental data in a pre-processing step
- Ideal situation
  - perform $M$ independent repeated experiments
  $$
  Y^{[m]}(k) = Y_0(k) + N_Y^{[m]}(k), \quad m = 1, 2, \ldots, M, \text{ and } k = 1, 2, \ldots, F
  $$
  $$
  U^{[m]}(k) = U_0(k) + N_U^{[m]}(k)
  $$
  where $N_U^{[m]}(k), N_Y^{[m]}(k)$ are independent (over $k$ and $m$) jointly correlated circular complex normally distributed disturbances
  - calculate sample means and sample (co-)variances $\hat{\sigma}_Y^2(k), \hat{\sigma}_U^2(k), \hat{\sigma}_{YU}^2(k)$
- Practise
  - $M$ consecutive periods of the steady state response to a multisine excitation
  - input/output disturbances $=$ filtered white noise
  - independence (over $k$ and $m$) and normality are asymptotically valid for $N \to \infty$ with $M$ fixed
6. Estimation parametric plant model with estimated nonparametric noise model

6.b Classical approach: sample maximum likelihood

- Use in ML cost function sample means and sample (co-)variances ⇒ “sample” ML

\[
V_{SML}(\theta, Z) = \sum_{k=1}^{F} \left\{ \frac{\hat{\varepsilon}(\Omega_k, \theta, \hat{Z}(k))}{\hat{\sigma}_\varepsilon^2(\Omega_k, \theta)} \right\}^2
\]

with

\[
\begin{align*}
\hat{\varepsilon}(\Omega_k, \theta, Z(k)) &= \hat{Y}(k) - G(\Omega_k, \theta) \hat{U}(k) \\
\hat{\sigma}_\varepsilon^2(\Omega_k, \theta) &= \hat{\sigma}_Y^2(k) + |G(\Omega_k, \theta)|^2 \hat{\sigma}_U^2(k) - 2 \text{Re}(\overline{G}(\Omega_k, \theta) \hat{\sigma}_Y \hat{\sigma}_U(k))
\end{align*}
\]

- Questions
  - minimal value \( M \)?
  - influence on the asymptotic properties of \( \hat{\theta}_{SML} \)?

- Key property to analyze the asymptotic behaviour of \( \hat{\theta}_{SML} \)
  - for \textit{normally} distributed errors the sample mean and sample covariance matrix are independently distributed
6. Estimation parametric plant model with estimated nonparametric noise model

6.b Classical approach: sample maximum likelihood (cont’d)

Asymptotic properties

• Consistency \( (M \geq 4) \)

\[
V_{\text{SML}}(\theta) = \mathbb{E} \{ V_{\text{SML}}(\theta, Z) \} = \sum_{k=1}^{F} \mathbb{E} \left\{ \left| \hat{e}(\Omega_k, \theta, \hat{Z}(k)) \right|^2 \frac{\hat{\sigma}_e^2(\Omega_k, \theta)}{\hat{\sigma}_e^2(\Omega_k, \theta)} \right\}
\]

\[
= \sum_{k=1}^{F} \mathbb{E} \left\{ \left| \hat{e}(\Omega_k, \theta, \hat{Z}(k)) \right|^2 \right\} \mathbb{E} \left\{ \frac{1}{\hat{\sigma}_e^2(\Omega_k, \theta)} \right\}
\]

\[
= \frac{M - 1}{M - 2} V_{\text{ML}}(\theta)
\]

⇒ minimal in \( \theta_0 \) ⇒ consistent

• Convergence rate, systematic and stochastic errors, asymptotic normality \( (M \geq 7) \)

same as ML (see section 5.b on page 221) with

\[
\text{Cov}(\delta_{\theta_{\text{SML}}}(Z)) \approx \frac{M - 2}{M - 3} \text{Cov}(\delta_{\theta_{\text{ML}}}(Z))
\]

⇒ small loss in efficiency (below 12% for \( M \geq 7 \))
6. Estimation parametric plant model with estimated nonparametric noise model

6.b Classical approach: sample maximum likelihood (cont’d)

Asymptotic properties (cont’d)

• Calculation covariance matrix

Fix a coefficient, e.g. \( a_{n_a} \), and remove the corresponding column in \( J_{\text{SML}} \)

\[
\text{Cov}(\hat{\theta}_{\text{SML}}(Z)) \approx \frac{M-1}{M-3} [2 \text{Re}(J^H_{\text{SML}}J_{\text{SML}})]^{-1} = \frac{M-1}{2(M-3)} (V \Sigma^{-1})(V \Sigma^{-1})^T
\]

with

\[
(J_{\text{SML}})_{[k, r]} = \frac{\partial \hat{e}_{[k]}}{\partial \theta_{\text{SML}[r]}}
\]

\[
\hat{e}_{[k]} = \frac{\hat{e}(\Omega_k, \theta, \hat{Z}(k))}{\hat{\sigma}_e(\Omega_k, \theta)}
\]

\[
\begin{bmatrix}
\text{Re}(J_{\text{SML}}) \\
\text{Im}(J_{\text{SML}})
\end{bmatrix} = U \Sigma V^T
\]

Note: perform the calculations on the frequency normalized parameters for \( s \)- and \( \sqrt{s} \)-domains
6. Estimation parametric plant model with estimated nonparametric noise model

6.b Classical approach: sample maximum likelihood (cont’d)

Asymptotic properties (cont’d)

• Analysis cost function

No modeling errors

\[
\begin{align*}
\mathbb{E}\{V_{\text{SML}}(\hat{\theta}_{\text{SML}}(Z), Z)\} & \approx \frac{M - 1}{M - 2} V_{\text{noise}} \\
\text{var}(V_{\text{SML}}(\hat{\theta}_{\text{SML}}(Z), Z)) & \approx \frac{(M - 1)^3}{(M - 2)^2(M - 3)} V_{\text{noise}}
\end{align*}
\]

with \( V_{\text{noise}} = F - n_\theta/2 \) and \( n_\theta = \text{number of free parameters} \)
6. Estimation parametric plant model with estimated nonparametric noise model

6.b Classical approach: sample maximum likelihood (cont’d)

Asymptotic properties (cont’d)

• Sample correlation residuals (Pintelon et al., 2003)

\[
\hat{R}_{\hat{\delta}\hat{\delta}}(m) = \frac{\alpha_1(m)}{F - m} \sum_{k=1}^{F-m} \hat{\delta}(\Omega_k, \hat{\theta}) \hat{\delta}(\Omega_{k+m}, \hat{\theta})
\]

with \( \hat{\delta}(\Omega_k, \hat{\theta}) = \frac{\hat{G}(\Omega_k) - G(\Omega_k, \hat{\theta})}{\hat{\sigma}\hat{G}(\Omega_k)} \)

\[
\alpha_1(m) = \begin{cases} 
(M - 2)/(M - 1) & m = 0 \\
(M - 5/3)/(M - 11/12) & m \neq 0 
\end{cases}
\]

Notes:
- 95% confidence bound \( \alpha_2(m) \sqrt{3/(F - m)} \) with

\[
\alpha_2(m) = \begin{cases} 
\alpha_1(m) \sqrt{3}(M - 1)^{3/2}/((M - 2)(M - 3)^{1/2}) & m = 0 \\
\alpha_1(m)(M - 1)/(M - 2) & m \neq 0 
\end{cases}
\]

- \( \hat{R}_{\hat{\delta}\hat{\delta}}(m) \) has exactly the same properties as \( \hat{R}_{\hat{\delta}\hat{\delta}}(m) \) in section 5.i on page 235
- \( \alpha_i(m) \) compensates for replacing the true variance by the sample variance
6. Estimation parametric plant model with estimated nonparametric noise model

6.c Measurement example: identification of Young’s modulus

\[
0 \leq t \leq L = 1,983 \text{ m}
\]

\[
\begin{align*}
M &= 25, \quad P = 10 \\
\text{measured FRF} &\quad \text{total variance} & \text{noise variance}
\end{align*}
\]

plexiglass beam

\[
1 \text{ cm} \times 2 \text{ cm} \times 1.983 \text{ m}
\]
6. Estimation parametric plant model with estimated nonparametric noise model

6.c Measurement example: identification of Young’s modulus (cont’d)

CT-model \( n_a = n_b = 34 \)

\[
V_{\text{SML}}(\hat{\theta}, Z) = 13023 \\
V_{\text{noise}} = 1602
\]
6. Estimation parametric plant model with estimated nonparametric noise model

6.c Measurement example: identification of Young’s modulus (cont’d)

\[ E(s_k) = -\rho \left( \frac{Ls_k}{k\pi} \right)^2 \left[ 1 + \left( \frac{k\pi \nu(s_k)J}{L} \right)^2 \right] \text{ for } k = 1, 2, \ldots, 7 \quad (\nu(s_k) \approx 0.3) \]

identified model of order \( n_a = n_b = 2 \)

measured Young’s modulus

\( V_{\text{ML}}(\hat{\theta}, Z) = 6.24 \times 10^4 \)

\( V_{\text{noise}} = 4.5 \)
6. Estimation parametric plant model with estimated nonparametric noise model

6.d Measurement example: Magnetic Resonance Imaging (MRI)

MRI scanner:

- \sim{} Tesla static magnetic field,
- \sim{} MHz oscillating field (pulse) perpendicular to the static field
- response measured in two orthogonal directions $x$ and $y$
  \Rightarrow{} complex signal $x(t) + jy(t)$
6. Estimation parametric plant model with estimated nonparametric noise model

6.d Measurement example: Magnetic Resonance Imaging (MRI) (con’t)

absolute value demodulated signal $x(t) + jy(t)$
(averaged over $M = 64$ independent measurements)
6. Estimation parametric plant model with estimated nonparametric noise model

6.d Measurement example: Magnetic Resonance Imaging (MRI) (con’t)

NMR spectrum muscle

signal model = sum of complex damped exponentials

\[ T(z^{-1}, \theta) = \frac{\sum_{r=0}^{n-1} b_r z^{-r}}{\sum_{r=0}^{n} a_r z^{-r}} \]

\( a_r, b_r \in \mathbb{C} \)

\( n = 9 \)
6. Estimation parametric plant model with estimated nonparametric noise model

6.d Measurement example: Magnetic Resonance Imaging (MRI) (con’t)

![Whiteness test residuals graph]

- Autocorrelation
  - 50% uncertainty bound (fraction outside = 51.6%)
  - 95% uncertainty bound (fraction outside = 5.2%)

\[ V_{\text{SML}}(\hat{\theta}, Z) = 584 \]
\[ V_{\text{noise}} = 502 \]
\[ V_{\text{noise}}^{1/2} = 22 \]
6. Estimation parametric plant model with estimated nonparametric noise model

6.d Measurement example: Magnetic Resonance Imaging (MRI) (con’t)

- frequency and damping peak depend on metabolite
- amplitude peak is a measure of the concentration of the metabolite
- concentration of a metabolite (e.g. creatine) in a colour scale as a function of space ⇒ MRI image

MRI brains
6. Estimation parametric plant model with estimated nonparametric noise model

6.e Classical approach: problems

Drawbacks

• periodic steady state only

• consecutive periods are correlated due to the noise transients ⇒ sample means and sample (co-)variances are not independently distributed

• sensitive to noise transient (leakage) errors

• sensitive to system transient (leakage) errors

Solution

• the local polynomial method (reason: transient suppression)
6. Estimation parametric plant model with estimated nonparametric noise model

6.f Local polynomial method: basic idea

Arbitrary excitations within an generalised output error framework

- Step 1: local polynomial estimates (see p. 164)
  - FRF: $\hat{G}(\Omega_k)$
  - noise covariance: $\hat{C}_V(k)$ with degrees of freedom $dof$

- Step 2: generalised sample mean and sample covariance (leakage free!)
  - sample mean: $\hat{Y}(k) = \hat{G}(\Omega_k)U(k)$
  - sample covariance sample mean: $\hat{C}_\hat{Y}(k) = \|q_n\|^2\hat{C}_V(k)$

where

$$q_n = K_n^H(K_nK_n^H)^{-1}\begin{bmatrix} U(k) \\ 0 \end{bmatrix}$$

with $K_n$ defined on p. 166

Note: errors-in-variables and periodic excitations: follow the same lines
6. Estimation parametric plant model with estimated nonparametric noise model

6.g Local polynomial method: MIMO sample maximum likelihood

- SML cost function

\[
V_{\text{SML}}(\theta, Z) = \sum_{k=1}^{F} \hat{e}^H(\Omega_k, \theta, \hat{Z}(k)) \hat{C}_{\hat{e}}^{-1}(k) \hat{e}(\Omega_k, \theta, \hat{Z}(k))
\]

with (no transient term!)

\[
\begin{aligned}
\hat{e}(\Omega_k, \theta, \hat{Z}(k)) &= \hat{Y}(k) - G(\Omega_k, \theta) \hat{U}(k) \\
\hat{C}_{\hat{e}}(\Omega_k, \theta) &= \hat{C}_{\hat{Y}}(k) + G(\Omega_k, \theta) \hat{C}_{\hat{U}}(k) G^H(\Omega_k, \theta) - 2 \text{herm}(\hat{C}_{\hat{Y}}\hat{U}(k) G^H(\Omega_k, \theta))
\end{aligned}
\]

- Question
  - minimal value \(dof\)?
  - influence on the asymptotic properties of \(\hat{\theta}_{\text{SML}}\)?

- Key property to analyze the asymptotic behaviour of \(\hat{\theta}_{\text{SML}}\)
  - the generalised sample mean and sample covariance are asymptotically independently distributed
6. Estimation parametric plant model with estimated nonparametric noise model

6.g Local polynomial method: MIMO sample maximum likelihood (cont’d)

Asymptotic properties

\[
\text{Cov}(\hat{\theta}_{\text{SML}}) = \frac{\text{dof} \ (\text{dof} - n_y)}{(\text{dof} - n_y + 1)(\text{dof} - n_y - 1)} \text{Cov}(\hat{\theta}_{\text{ML}}) \quad (\text{dof} \geq n_y + 8)
\]

\[
\mathbb{E}\{V_{\text{SML}}(\hat{\theta}_{\text{SML}}, Z)\} \approx \frac{\text{dof}}{\text{dof} - n_y} V_{\text{noise}} \quad V_{\text{noise}} = n_y F - n_\theta/2 \quad (\text{dof} \geq n_y + 2)
\]

\[
\sigma_1^2 \leq \text{var}(V_{\text{SML}}(\hat{\theta}_{\text{SML}}, Z)) \leq 3 \sigma_1^2 \quad \sigma_1^2 = \frac{\text{dof}^3}{(\text{dof} - n_y)^2(\text{dof} - n_y - 1)} V_{\text{noise}} \quad (\text{dof} \geq n_y + 4)
\]

\[
\text{Cov}(\hat{\theta}_{\text{SML}}) \approx \frac{\text{dof}^2}{(\text{dof} - n_y + 1)(\text{dof} - n_y - 1)} (2 \text{Re}(J_{\text{SML+}}^H J_{\text{SML+}}))^{-1} \quad (\text{dof} \geq n_y + 8)
\]

Notes:

- pseudo jacobian \( J_{\text{SML+}} \) (see p. 237) with \( a_{n_a} \) removed

- classical approach: \( M = \text{dof} + 1 \)

- whiteness test residuals (see p. 235), where \( M = \text{dof} + 1 \), is valid over a subset of frequencies (see p. 168, correlation length LPM estimates)
6. Estimation parametric plant model with estimated nonparametric noise model

6.h MIMO experiment: modal analysis Aluminium tooling plate

size Al plate: 30.4 cm x 61.8 cm x 6.7 mm
6. Estimation parametric plant model with estimated nonparametric noise model

6.h MIMO experiment: modal analysis Aluminium tooling plate (cont’d)

Measurement

- two consecutive periods of the transient response to 1 set of $n_u = 2$ uncorrelated random phase multisines
- $f_s = 2.44 \text{ kHz}, \; N = 64 \times 1024$ points per period

Local polynomial method for periodic excitations

- fast method with $R = 4$ and $dof = 9 \Rightarrow n = 9$
- correlation length nonparametric estimates is $\pm 18$

Parametric model

- continuous-time common denominator model
- model order selection via MDL and AIC (see p. 239)
6. Estimation parametric plant model with estimated nonparametric noise model

6.h MIMO experiment: modal analysis Aluminium tooling plate (cont’d)

<table>
<thead>
<tr>
<th>$n_b/n_a$</th>
<th>$V_{SML}$</th>
<th>$\mathbb{E}{V_{SML}}$</th>
<th>std($V_{SML}$)</th>
<th>MDL</th>
<th>AIC</th>
</tr>
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<tbody>
<tr>
<td>6/4</td>
<td>9855</td>
<td>462.5</td>
<td>[29.9, 51.7]</td>
<td>12920</td>
<td>10690</td>
</tr>
<tr>
<td>6/5</td>
<td>9828</td>
<td>462.2</td>
<td>[29.9, 51.7]</td>
<td>12980</td>
<td>10690</td>
</tr>
<tr>
<td>6/6</td>
<td>527.4</td>
<td>461.6</td>
<td>[29.8, 51.7]</td>
<td>701.9</td>
<td>575.1</td>
</tr>
<tr>
<td>7/6</td>
<td>394.4</td>
<td>459.0</td>
<td>[29.7, 51.5]</td>
<td>540.3</td>
<td>434.4</td>
</tr>
<tr>
<td>8/6</td>
<td>365.2</td>
<td>456.4</td>
<td>[29.7, 51.4]</td>
<td>514.5</td>
<td>406.0</td>
</tr>
<tr>
<td>8/7</td>
<td>363.0</td>
<td>455.8</td>
<td>[29.6, 51.4]</td>
<td>515.0</td>
<td>404.6</td>
</tr>
</tbody>
</table>

MDL criterion: $n_b/n_a = 8/6$

Parsimonious principle: $n_b/n_a = 7/6$
6. Estimation parametric plant model with estimated nonparametric noise model

6.h MIMO experiment: modal analysis Aluminium tooling plate (cont’d)

\[ n_b / n_a = 8 / 6 \]
6. Estimation parametric plant model with estimated nonparametric noise model

6.h MIMO experiment: modal analysis Aluminium tooling plate (cont’d)

![Graphs showing angle and phase plots for different frequency responses.](image-url)
6. Estimation parametric plant model with estimated nonparametric noise model

6.h MIMO experiment: modal analysis Aluminium tooling plate (cont’d)
6. Estimation parametric plant model with estimated nonparametric noise model

6.h MIMO experiment: modal analysis Aluminium tooling plate (cont’d)

Modal parameters for the MDL (8/6) and parsimonious (7/6) models

<table>
<thead>
<tr>
<th></th>
<th>( n_b/n_a = 8/6 )</th>
<th>( n_b/n_a = 7/6 )</th>
<th>Difference Estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_0 \pm \text{std}(f_0) ) (Hz)</td>
<td>( 250.8552 \pm 2.0 \times 10^{-5} )</td>
<td>( 250.8550 \pm 2.0 \times 10^{-5} )</td>
<td>( 2.0 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \zeta \pm \text{std}(\zeta) )</td>
<td>( 2.96 \times 10^{-4} \pm 3.2 \times 10^{-6} )</td>
<td>( 2.97 \times 10^{-4} \pm 3.1 \times 10^{-6} )</td>
<td>( -1.4 \times 10^{-6} )</td>
</tr>
<tr>
<td>( f_0 \pm \text{std}(f_0) ) (Hz)</td>
<td>( 251.7844 \pm 3.8 \times 10^{-5} )</td>
<td>( 251.7840 \pm 3.7 \times 10^{-5} )</td>
<td>( 4.0 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \zeta \pm \text{std}(\zeta) )</td>
<td>( 9.0 \times 10^{-5} \pm 6.0 \times 10^{-6} )</td>
<td>( 9.2 \times 10^{-5} \pm 5.9 \times 10^{-6} )</td>
<td>( -1.5 \times 10^{-6} )</td>
</tr>
</tbody>
</table>
6. Estimation parametric plant model with estimated nonparametric noise model

6.i References
7. Estimation parametric plant and/or noise models

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<td>7.i Measurement example: flight flutter - MIMO case (cont’d)</td>
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</table>
7. Estimation parametric plant and/or noise models

7.a Classical time domain prediction error framework

- Discrete-time plant and noise model

Model

\[ y(t) = G(q)u(t) + H(q)e(t) \]

with a monic noise model

\[ H(z^{-1}) = \frac{1 + c_1 z^{-1} + \ldots}{1 + d_1 z^{-1} + \ldots} \]

Notes:

- \( u(t) \) and \( y(t) \) are known exactly and \( e(t) \) is not observed
- \( e(t) \) is not observed \( \Rightarrow \) \( H(z^{-1}) \) is monic by appropriate scaling
- \( H(z^{-1}) \) and \( H^{-1}(z^{-1}) \) should be stable
7. Estimation parametric plant and/or noise models

7.a Classical time domain prediction error framework (cont’d)

- One-step-ahead prediction

\[ v(t) = H(q)e(t) = e(t) + (H(q) - 1)e(t) = e(t) + (1 - H^{-1}(q))v(t) \]

One-step-ahead predictor process noise

\[ \hat{v}(t|t-1) = (1 - H^{-1}(q))v(t) \]

= best (in least square sense) estimate of \( v(t) \) based on the knowledge of all past samples \( v(t-1), v(t-2), \ldots, v(-\infty) \)

One-step-ahead predictor output

\[ \hat{y}(t|t-1) = G(q)u(t) + \hat{v}(t|t-1) = G(q)u(t) + (1 - H^{-1}(q))(y(t) - G(q)u(t)) \]

Prediction error

\[ \varepsilon(t) = y(t) - \hat{y}(t|t-1) \]

\[ = y(t) - G(q)u(t) - (1 - H^{-1}(q))(y(t) - G(q)u(t)) \]

\[ = H^{-1}(q)(y(t) - G(q)u(t)) \]
7. Estimation parametric plant and/or noise models

7.a Classical time domain prediction error framework (cont’d)

• Prediction error estimator

Prediction error cost function

\[ V_{PE}(\theta, z) = \sum_{t=0}^{N-1} \epsilon^2(t, \theta) = \sum_{t=0}^{N-1} (H^{-1}(q, \theta)(y(t) - G(q, \theta)u(t)))^2 \]

Notes:
- the effect of the initial conditions is hidden in the filter operations
- the predictor should be stable, viz., \( H^{-1} \) and \( H^{-1}G \)
- according to the particular model structure one distinguishes
  
  ARX: \( C = 1 \), and \( D = A \) ⇒ linear least squares
  ARMAX: \( D = A \)
  ARMA: \( G = 0 \) ⇒ time series/spectral analysis
  OE: \( H = 1 \)
  BJ: \( G \) and \( H \) are independently parametrized
- frequency domain equivalent (Parseval’s theorem)

\[ \sum_{t=0}^{N-1} \epsilon^2(t, \theta) = \sum_{k=0}^{N-1} |\epsilon(z_k^{-1}, \theta)|^2 \]

\[ \epsilon(z_k^{-1}, \theta) = H^{-1}(z_k^{-1}, \theta)(Y(k) - G(z_k^{-1}, \theta)U(k) - T_G(z_k, \theta) - T_H(z_k, \theta)) \]
7. Estimation parametric plant and/or noise models

7.a Classical time domain prediction error framework (cont’d)

- Properties estimated plant model parameters

  Consistent
  
  Open loop
  
  OE, BJ: even with wrong noise model structure
  ARX, ARMAX: correct noise model structure
  
  Closed loop
  
  OE, BJ, ARX, ARMAX: correct noise model structure

  Asymptotically normally distributed

  Maximum likelihood for normally distributed noise

  Asymptotically efficient for normally distributed noise

  Cramér-Rao lower bound: see Ljung (1999)
7. Estimation parametric plant and/or noise models

7.a Classical time domain prediction error framework (cont’d)

• Focusing on a particular frequency band - goal

Observation
- the frequency content of a sampled signal extends from DC to Nyquist

Goal
- removal slow trends and/or high frequency disturbances
- removal (harmonic) disturbances that cannot be written as filtered white noise
- simplified plant and/or noise model
7. Estimation parametric plant and/or noise models

7.a Classical time domain prediction error framework (cont’d)

- Focusing on a particular frequency band - time domain

\[ y(t) = G(q)u(t) + H(q)e(t) \]

\[ y_f(t) = G(q)u_f(t) + F(q)H(q)e(t) \]

\[ \varepsilon_f(t) = (F(q)H(q))^{-1}(y_f(t) - G(q)u_f(t)) = H^{-1}(q)(y(t) - G(q)u(t)) = \varepsilon(t) \]
7. Estimation parametric plant and/or noise models

7.a Classical time domain prediction error framework (cont’d)

- Focusing on a particular frequency band - time domain

Properties

- input/output relation is preserved
- prefiltering is equivalent to dividing the noise model by the prefilter

Requirement

- noise model should be flexible enough to preserve efficiency (open/closed loop) and consistency (closed loop)

Conclusion

- classical time domain solution = compromise between suppression undesired frequency bands and loss in efficiency and consistency
7. Estimation parametric plant and/or noise models

7.1 Classical time domain prediction error framework (cont’d)

- Focusing on a particular frequency band - frequency domain

Properties
- exact non-causal filtering
- no loss in consistency
- increased variance
7. Estimation parametric plant and/or noise models

7.b Frequency domain identification in closed loop with known controller

\[ V(k) = H(\Omega_k)E(k) \]

Closed loop equations

\[
\begin{align*}
U(k) &= \frac{1}{1 + G(\Omega_k)M_0(\Omega_k)}R(k) - \frac{M_0(\Omega_k)H(\Omega_k)}{1 + G(\Omega_k)M_0(\Omega_k)}E(k) \\
Y(k) &= \frac{G(\Omega_k)}{1 + G(\Omega_k)M_0(\Omega_k)}R(k) + \frac{H(\Omega_k)}{1 + G(\Omega_k)M_0(\Omega_k)}E(k)
\end{align*}
\]

Conditional expectations (\( \lambda = \text{var}(E(k)) \))

\[
\begin{align*}
\mathbb{E}\{ Y(k) | R(k), \theta, \lambda \} &= \frac{G(\Omega_k, \theta)}{1 + G(\Omega_k, \theta)M_0(\Omega_k)}R(k) \equiv Y(k, \theta) \\
\text{var}(Y(k) | R(k), \theta, \lambda) &= \text{var}\left( \frac{H(\Omega_k, \theta)}{1 + G(\Omega_k, \theta)M_0(\Omega_k)}E(k) \right) = \lambda |S(\Omega_k, \theta)|^2
\end{align*}
\]
7. Estimation parametric plant and/or noise models

7.b Frequency domain identification in closed loop with known controller (cont’d)

- Gaussian ML estimator

Gaussian likelihood function of $Y(k)$

$$f_{Y(k)}(Y(k)|R(k), \theta, \lambda) = \frac{1}{\pi\lambda|S(\Omega_k, \theta)|^2} \exp\left(-\frac{|Y(k) - Y(k, \theta)|^2}{\lambda|S(\Omega_k, \theta)|^2}\right)$$

Negative Gaussian log-likelihood of all data

$$-\sum_{k \in \mathbb{K}} \log(f_{Y(k)}(Y(k)|R(k), \theta, \lambda)) = C + \sum_{k \in \mathbb{K}} \log(\lambda|S(\Omega_k, \theta)|^2) + \sum_{k \in \mathbb{K}} \frac{|\epsilon(\Omega_k, \theta)|^2}{\lambda}$$

Eliminating $\lambda$ gives

$$V_F(\theta, Z) = \frac{1}{F} \sum_{k \in \mathbb{K}} |\epsilon(\Omega_k, \theta)g_F(\theta)|^2 \text{ with } g_F(\theta) = \exp\left(\frac{1}{F} \sum_{k \in \mathbb{K}} \log\left(\frac{H(\Omega_k, \theta)}{1 + G(\Omega_k, \theta)M_0(\Omega_k)}\right)\right)$$

Notes:

- open loop $M_0 = 0$ (Ljung, 1999)
- knowledge controller necessary (McKelvey, 2000) ⇒ besides $u(t)$, $y(t)$ store also $r(t)$
- numerically stable Newton-Gauss scheme of the form $J\Delta \theta = -e$
7. Estimation parametric plant and/or noise models

7.b Frequency domain identification in closed loop with known controller (cont’d)

• Improving the finite sample behaviour

\[ V_F(\theta, Z) = \frac{1}{F} \sum_{k \in K} |\epsilon(\Omega_k, \theta)g_F(\theta)|^2 \text{ with } g_F(\theta) = \exp\left(\frac{1}{F} \sum_{k \in K} \log(S(\Omega_k, \theta))\right) \]

with prediction error

\[ \epsilon(\Omega_k, \theta) = \frac{Y(k) - G(\Omega_k, \theta)U(k) - T_G(\Omega_k, \theta) - T_H(\Omega_k, \theta)}{H(\Omega_k, \theta)} \]

• Model structures

DT-ARX, CT-ARX: \( C = 1 \), \( D = A \), and \( T_H = 0 \) \( \Rightarrow \) nonlinear least squares

DT-ARMAX, CT-ARMAX: \( D = A \), \( T_H = 0 \), and \( n_{ig} \geq \max(n_a, n_b, n_c) - 1 \)

DT-ARMA, CT-ARMA: \( G = 0 \), and \( T_G = 0 \) \( \Rightarrow \) time series/spectral analysis

DT-OE, CT-OE: \( H = 1 \), and \( T_H = 0 \)

DT-BJ, CT-BJ: \( G \) and \( H \) are independently parametrized

Hybrid-BJ: CT-plant and DT-noise models
7. Estimation parametric plant and/or noise models

7.b Frequency domain identification in closed loop with *known* controller (cont’d)

- Connection with the classical prediction error framework

\[ V_F(\theta, Z) = \frac{1}{F} \sum_{k \in \mathbb{K}} |\varepsilon(z_k^{-1}, \theta)g_F(\theta)|^2 \]

If

1. \[ \lim_{z \to \infty} S(z^{-1}, \theta) = 1 \]
2. \( \mathbb{K} \) covers uniformly the unit circle

then

\[ g_F(\theta) = \exp\left(\frac{1}{N} \sum_{k=0}^{N-1} \log(S(z_k^{-1}, \theta))\right) = 1 + O(|\lambda_{\text{max}}|^N) \]

with \( \lambda_{\text{max}} \) the dominant pole of

\[ \frac{d}{dz} \log(S(z^{-1}, \theta)) = \frac{d}{dz} \log\left(\frac{H(z^{-1}, \theta)}{1 + G(z^{-1}, \theta)M_0(z^{-1})}\right) \]

Notes:

- condition 1 is fulfilled if
  - monic noise model
  - at least one sample delay in plant and/or controller
- knowledge controller no longer necessary
- \( g_F(\theta) = 1 \Rightarrow \) classical prediction error method
7. Estimation parametric plant and/or noise models

7.c Frequency domain identification in closed loop with unknown controller

\[ V(k) = H(\Omega_k)E(k) \]

\[ R(k) = L(\Omega_k)W(k) \]

\[ U(k) \]

\[ G(\Omega_k) \]

\[ M(\Omega_k) \]

\[ Y(k) \]

\[ \varepsilon_G(\Omega_k, \theta) = \frac{Y(k) - G(\Omega_k, \theta)U(k)}{H(\Omega_k, \theta)} \]

\[ \varepsilon_M(\Omega_k, \theta) = \frac{U(k) + M(\Omega_k, \theta)Y(k)}{L(\Omega_k, \theta)} \]

- Gaussian ML estimator

Gaussian likelihood function of \( Z(k) = [Y(k), U(k)]^T \) (\( \lambda = \text{var}(E(k)), \mu = \text{var}(W(k)) \))

\[ \mathbb{E} \{ Z(k) | \theta, \lambda, \mu \} = 0 \]

\[ C_{Z(k)} = \mathbb{E} \{ Z^H(k)Z(k) | \theta, \lambda, \mu \} = \frac{1}{|1 + GM|^2} \left[ \frac{\mu|GL|^2 + \lambda|H|^2}{\mu G |L|^2 - \lambda M |H|^2} \right] \]

\[ f_{Z(k)}(Z(k) | \theta, \lambda, \mu) = \frac{1}{\pi^2 \det C_{Z(k)}} \exp(-Z^H(k)C_{Z(k)}^{-1}Z(k)) \]
7. Estimation parametric plant and/or noise models

7.c Frequency domain identification in closed loop with unknown controller (cont’d)

• Gaussian ML estimator (cont’d)

Negative Gaussian log-likelihood function all data

\[- \sum_{k \in K} \log(f_{Z(k)}(Z(k) | \theta, \lambda, \mu)) = C_1 + \sum_{k \in K} \log(\lambda \mu |T(\Omega_k, \theta)|^2) + \sum_{k \in K} \frac{|\varepsilon_G(\Omega_k, \theta)|^2}{\lambda} + \sum_{k \in K} \frac{|\varepsilon_M(\Omega_k, \theta)|^2}{\mu}\]

where

\[T(\Omega_k, \theta) = \frac{H(\Omega_k, \theta)L(\Omega_k, \theta)}{1 + G(\Omega_k, \theta)M(\Omega_k, \theta)}\]

Eliminating \(\lambda\) and \(\mu\) gives

\[V_F(\theta, Z) = \left|h_F(\theta)\right|^2 \left(\frac{1}{F} \sum_{k \in K} \left|\varepsilon_G(\Omega_k, \theta)\right|^2\right) \left(\frac{1}{F} \sum_{k \in K} \left|\varepsilon_M(\Omega_k, \theta)\right|^2\right)\]

\[h_F(\theta) = \exp\left(\frac{1}{F} \sum_{k \in K} \log(T(\Omega_k, \theta))\right)\]

Notes:

- symmetric in \(G, H\) and \(M, L\)
- identification \(G, H\) is coupled with that of \(M, L\) via \(T = HL/(1 + GM)\)
- numerically stable Newton-Gauss scheme of the form \(J \Delta \theta = -e\)
  (see Pintelon and Schoukens, 2006)
7. Estimation parametric plant and/or noise models

7.c Frequency domain identification in closed loop with unknown controller (cont’d)

• Improving the finite sample behaviour

\[ V_F(\theta, Z) = |h_F(\theta)|^2 \left( \frac{1}{F} \sum_{k \in K} |\varepsilon_G(\Omega_k, \theta)|^2 \right) \left( \frac{1}{F} \sum_{k \in K} |\varepsilon_M(\Omega_k, \theta)|^2 \right) \]

\[ h_F(\theta) = \exp \left( \frac{1}{F} \sum_{k \in K} \log(T(\Omega_k, \theta)) \right) \]

with plant and controller prediction errors

\[ \varepsilon_G(\Omega_k, \theta) = \frac{Y(k) - G(\Omega_k, \theta)U(k) - T_G(\Omega_k, \theta) - T_H(\Omega_k, \theta)}{H(\Omega_k, \theta)} \]

\[ \varepsilon_M(\Omega_k, \theta) = \frac{U(k) + M(\Omega_k, \theta)Y(k) + T_M(\Omega_k, \theta) - T_L(\Omega_k, \theta)}{L(\Omega_k, \theta)} \]
7. Estimation parametric plant and/or noise models

7.c Frequency domain identification in closed loop with unknown controller (cont’d)

• Connection with the time domain joint input-output approach

\[ V_F(\theta, Z) = |h_F(\theta)|^2 \left( \frac{1}{F} \sum_{k \in \mathbb{K}} |\varepsilon_G(\Omega_k, \theta)|^2 \right) \left( \frac{1}{F} \sum_{k \in \mathbb{K}} |\varepsilon_M(\Omega_k, \theta)|^2 \right) \]

If

1. \( \lim_{z \to \infty} T(z^{-1}, \theta) = 1 \)
2. \( \mathbb{K} \) covers uniformly the unit circle

then

\[ h_F(\theta) = \exp \left( \frac{1}{N} \sum_{k=0}^{N-1} \log (T(z^{-1}, \theta)) \right) = 1 + O(\lambda_{\text{max}} N/N) \]

with \( \lambda_{\text{max}} \) the dominant pole of \( \frac{d}{dz} \log (T(z^{-1}, \theta)) = \frac{d}{dz} \log \left( \frac{H(z^{-1}, \theta)L(z^{-1}, \theta)}{1 + G(z^{-1}, \theta)M(z^{-1}, \theta)} \right) \)

Notes:

- condition 1 is fulfilled if
  • monic noise and signal models
  • at least one sample delay in plant and/or controller
- identification of \( G, H \) is no longer coupled with that of \( M, L \)
- \( h_F(\theta) = 1 \Rightarrow \) joint input-output approach
  (Ljung, 1999; Söderström and Stoica, 1989)
7. Estimation parametric plant and/or noise models

7.d Measurement example: nonlinear electrical circuit

- White noise excitation with bandwidth of 200 Hz
- Measurement time 17 s: $N = 10400$ data samples at $f_s = 20 \text{ MHz}/2^{15} \approx 600 \text{ Hz}$
- CT-plant model in the band $[0 \text{ Hz}, 200 \text{ Hz}]$ with DT- or CT-noise model
7. Estimation parametric plant and/or noise models

7.d Measurement example: nonlinear electrical circuit (cont’d)

CT-plant model

\[ n_a = 2, \quad n_b = 0 \]

DT-noise model

\[ n_c = n_d = 8 \]

DT-noise model

\[ n_c = 6, \quad n_d = 5 \]

CT-noise model

\[ n_c = 4, \quad n_d = 5 \]

\[ \hat{V} = Y - G_{\dot{\theta}} U - T_{\dot{\theta}} \]

\[ \lambda_{\dot{\theta}}^{1/2} H_{\dot{\theta}} \]

\[ \text{var}(\lambda_{\dot{\theta}}^{1/2} H_{\dot{\theta}}) \]
7. Estimation parametric plant and/or noise models

7.e Measurement example: Villa Passo bridge

- Unobserved excitations: traffic + turbulent air flow
- Acceleration measurements in test points
- Measurement time 14 min: \( N = 337700 \) data points per channel at \( f_s = 400 \) Hz
- Model the band \([1.18 \, \text{Hz}, 4.14 \, \text{Hz}]\): DFT lines \( k = 999, 1000, \ldots, 3499 \) \( \Rightarrow F = 2501 \)
7. Estimation parametric plant and/or noise models

7.e Measurement example: Villa Passo bridge (cont’d)

[Graphs showing frequency response for test points 1 and 6 with CT-ARMA 8/8 models.]

\[ \hat{V} = Y - T_{\theta} \]
\[ \lambda_{\hat{\theta}}^{1/2} H_{\hat{\theta}} \]
\[ \text{var}(\lambda_{\hat{\theta}}^{1/2} H_{\hat{\theta}}) \]
7. Estimation parametric plant and/or noise models

7.f Measurement example: flight flutter

- Applied excitation: perturbation force at flap right wing
- Acceleration measurement at left wing
- Unobserved excitation: turbulent air flow
- Measurement time 109 s: \( N = 32768 \) data points per channel at \( f_s = 300 \) Hz
- Model the band \([4.70 \text{ Hz}, 16.40 \text{ Hz}]\): DFT lines \( k = 513, 514, \ldots, 1791 \) \( \Rightarrow \) \( F = 1279 \)
7. Estimation parametric plant and/or noise models

7.f Measurement example: flight flutter (cont’d)

CT-ARMAX
\[ n_a = 6, \ n_b = 8, \ n_c = 10 \]

CT-Box-Jenkins
\[ n_a = 6, \ n_b = 8, \ n_c = 10, \ n_d = 6 \]

\[
\begin{align*}
\hat{G}_{LPM} & \\
G_{\hat{\theta}} & \\
|\hat{G}_{LPM} - G_{\hat{\theta}}| & \\
\text{var}(\hat{G}_{LPM}) & \\
\text{var}(G_{\hat{\theta}}) &
\end{align*}
\]

\[
\begin{align*}
\hat{V} = Y - G_{\hat{\theta}}U & \\
- T_{\hat{\theta}} & \\
\lambda_{\hat{\theta}}^{1/2} H_{\hat{\theta}} & \\
\text{var}(\lambda_{\hat{\theta}}^{1/2} H_{\hat{\theta}}) &
\end{align*}
\]
7. Estimation parametric plant and/or noise models

7.g Multivariable systems in closed loop with known controller

\[
V(k) = H(\Omega_k)E(k)
\]

\[
\begin{array}{c}
R(k) + U(k) \\
\downarrow \\
G(\Omega_k) \\
\downarrow \\
Y(k) \\
\end{array}
\]

- Gaussian ML estimator

\[
- \sum_{k \in \mathcal{K}} \log(f_{Y(k)}(Y(k)|R(k), \theta, A)) = C_1 + \sum_{k \in \mathcal{K}} \log(|\det(S(\Omega_k, \theta))|^2) + F \log\det(A) + \ldots
\]

\[
\sum_{k \in \mathcal{K}} \varepsilon^H(\Omega_k, \theta) A^{-1} \varepsilon(\Omega_k, \theta)
\]

with

\[
\varepsilon(\Omega_k, \theta) = H^{-1}(\Omega_k, \theta)(Y(k) - G(\Omega_k, \theta)U(k))
\]

\[
S(\Omega_k, \theta) = (I_{n_y} + G(\Omega_k, \theta)M_0(\Omega_k))^{-1}H(\Omega_k, \theta)
\]

\[
\text{and } A = \mathbb{E}\{E(k)E^H(k)\} \in \mathbb{R}^{n_y \times n_y}
\]
7. Estimation parametric plant and/or noise models

7.g Multivariable systems in closed loop with known controller (cont’d)

- Gaussian ML estimator (cont’d)

  Eliminating \( \Lambda \) gives

\[
V_F(\theta, Z) = \det(\text{Re}(\frac{1}{F} \sum_{k \in \mathcal{K}} |g_F(\theta)|^2 \mathcal{E}(\Omega_k, \theta) \mathcal{E}^H(\Omega_k, \theta))) \quad \text{with} \quad g_F(\theta) = \exp\left(\frac{1}{n_y} \sum_{k \in \mathcal{K}} \log \det(S(\Omega_k, \theta))\right)
\]

and

\[
\mathcal{E}(\Omega_k, \theta) = H^{-1}(\Omega_k, \theta)(Y(k) - G(\Omega_k, \theta)U(k))
\]

\[
S(\Omega_k, \theta) = (I_{n_y} + G(\Omega_k, \theta)M_0(\Omega_k))^{-1}H(\Omega_k, \theta)
\]

Notes:

- no problems with calculation \( H^{-1}(\Omega, \theta) \)
- numerical stable Newton-Gauss scheme of the form \( J\Delta \theta = -e \)
  (see Pintelon et al., 2007)
7. Estimation parametric plant and/or noise models

7.g Multivariable systems in closed loop with known controller (cont’d)

- Asymptotic properties ML estimator

  Consistency

  Open loop
  - all model structures

  Closed loop
  - all model structures except Hybrid BJ

Asymptotically normally distributed

Asymptotically efficient

Robustness
  - consistency and asymptotic normality remain valid for non-Gaussian mixing frequency domain noise

Note:

- consistency plant model parameters **always** requires the correct noise model structure, which is not the case for the classical PE method (see section 7.a on page 273)
7. Estimation parametric plant and/or noise models

7.g Multivariable systems in closed loop with known controller (cont’d)

• Improving the finite sample behaviour

Prediction error

\[ \varepsilon(\Omega_k, \theta) = H^{-1}(\Omega_k, \theta)(Y(k) - G(\Omega_k, \theta)U(k) - T_G(\Omega_k, \theta) - T_H(\Omega_k, \theta)) \]

with

- \( T_G(\Omega_k, \theta) \) the plant transient term
- \( T_H(\Omega_k, \theta) \) the noise transient term

Common denominator parametrization

\[
\begin{align*}
G(\Omega, \theta) &= \frac{B(\Omega, \theta)}{A(\Omega, \theta)}, & T_G(\Omega_k, \theta) &= \frac{I_G(\Omega, \theta)}{A(\Omega, \theta)} \\
H(\Omega, \theta) &= \frac{C(\Omega, \theta)}{D(\Omega, \theta)}, & T_H(\Omega_k, \theta) &= \frac{I_H(\Omega, \theta)}{D(\Omega, \theta)}
\end{align*}
\]

where

- \( A \): polynomial, \( B \): \( n_y \times n_u \) matrix polynomial, \( I_G \): \( n_y \times 1 \) vector polynomial
- \( C \): \( n_y \times n_y \) matrix polynomial, \( D \): polynomial, \( I_H \): \( n_y \times 1 \) vector polynomial
7. Estimation parametric plant and/or noise models

7.g Model selection

Model selection criteria

\[
\left( \frac{2F}{2F - n_\theta} \right)^{n_y} V_F(\hat{\theta}(Z), Z) e^{p(n_\theta, F)}
\]

with ML cost function

\[
V_F(\theta, Z) = \det(\text{Re}\left( \frac{1}{F} \sum_{k \in \Omega} |g_F(\theta)|^2 \varepsilon(\Omega_k, \theta) \varepsilon^H(\Omega_k, \theta) \right))
\]

and penalty

\[
p(n_\theta, F) = \begin{cases} 
2(n_\theta + 1) & \text{AIC} \\
\frac{2(n_\theta + 1)}{2F - n_\theta - 2} & \text{MDL} \\
\log(2n_x F)(n_\theta + 1) & \end{cases}
\]

Notes:
- complex models are more likely to be rejected than in case of known noise models: compare \(e^{p(n_\theta, F)}\) with \((1 + p(n_\theta, F))\) (see section 5.1 on page 239)
- for very small data sets \(V_F(\hat{\theta}(Z), Z)e^{p(n_\theta, F)}\) may perform better for MDL
7. Estimation parametric plant and/or noise models

7.h Measurement example: Villa Passo bridge - MIMO case

CT-ARMA

\[ n_c = 10, \ n_d = 8 \]

- \[ \hat{V} \hat{V}^H \]
- \[ H_\theta \Lambda_\theta H_\theta^H \]
- \[ \text{var}(H_\theta \Lambda_\theta^{1/2}) \]

\[ \hat{V} = Y - T_\theta \]

\[ V \]
7. Estimation parametric plant and/or noise models

7.i Measurement example: flight flutter - MIMO case

CT-ARMAX

\[ n_a = 8, n_b = 10, n_c = 8 \]
7. Estimation parametric plant and/or noise models

7.i Measurement example: flight flutter - MIMO case (cont’d)

CT-ARMAX

\[ n_a = 8, n_b = 10, n_c = 8 \]

\[ \hat{V} = Y - G_\theta U - T_\theta \]
7. Estimation parametric plant and/or noise models

7.j References

*Generalized output error*


*Errors-in-variables*


8. Guidelines

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8. Guidelines

8.a Two basic questions

Stochastic framework?
- Output observations only
- Known input - output disturbed by unobserved input(s)
- Errors-in-variables
- Output error - indep. plant and noise dyn.

Noise model
- Parametric
- Nonparametric

Identification method
- Prediction error
- Frequency domain maximum likelihood
- Sample maximum likelihood
8. Guidelines

8.a Two basic questions (cont’d)

**Application?**
- Digital control
- Physical interpretation
  - Analog simulation
- Digital simulation

**Exp. setup**
- Zero-order-hold
- Band-limited
- Band-limited

**Domain**
- Discrete-time
- Continuous-time
- Continuous-time
8. Guidelines

8.b Identification step by step

- Check and selection of the experimental setup
- Design of an experiment
- Choice noise model
- Preprocessing of the data
- The identification step
8. Guidelines

8.b Identification step by step (cont’d)

Check and selection of the experimental setup

- Visit the site of the experimental setup and talk to the operators to learn from their experience
- Check the systematic errors of the complete data acquisition - calibrate the setup
- Check the validity of the signal assumptions (ZOH or BL, anti-alias filters)
- Pay attention to the synchronisation of the setup
- Always save the signal stored in the arbitrary waveform generator together with the noisy input-output data
8. Guidelines

8.b Identification step by step (cont’d)

Design of an experiment

- Choose the excitation (random or periodic) that best fits your specific needs
- Select the amplitude range and frequency band of the excitation signal to cover the frequency band of interest
- Check for the presence of nonlinear distortions and the time-variant effects
- Check whether the device is captured in a feedback loop
- Keep your application in mind
8. Guidelines

8.b Identification step by step (cont’d)

Choice noise model

• Default choice: nonparametric models

• Use parametric noise models for the following problems:
  - output data only
  - known input / output disturbed by unobserved random input(s)
8. Guidelines

8.b Identification step by step (cont’d)

Preprocessing of the data

- Removal of trends, drifts, and offsets
  - Check for trends by calculating the mean value over the successive periods
  - Do not use the DC information during the identification

- Dealing with outliers and missing data
  - Perform new experiments
  - If this is not possible, estimate first nonparametrically the FRF and the missing data. Next, use this as starting value for the parametric estimation of the plant model in the presence of missing data

- Check whether the system is linear and time invariant
  - Check the presence of non-excited harmonics in the DFT spectra
  - Check the presence of skirts in the output DFT spectrum of all samples

- Estimate the nonparametric FRF
  - Calculate the FRF + noise level + level nonlinear distortions + level time-variant effects
  - Select the frequency band of interest
8. Guidelines

8.b Identification step by step (cont’d)

The identification step

• Choice of a model class

  - s-, z- or √s- domain

• Model selection/ validation

  - Compare the transfer function model to the nonparametric FRF estimate
  - Analyse the value of the cost function
  - Perform a whiteness test of the residuals
  - Detect overmodeling via AIC (prediction) or MDL (physical interpretation)

• Impact of transient/leakage errors

  - Suppress nonparametrically the transient (leakage) errors in the data
  - If the transient time can be neglected, measure the steady state response
  - Always add the plant and noise transient terms when estimating parametric noise models
  - Suppress (non)parametrically the transient (leakage) errors in the validation data
8. Guidelines

8.1 References


9. Take Home Messages

1. Always check and calibrate the measurement setup

2. Feedback is almost always present in a band-limited setup (interaction non-ideal actuator - plant)

3. Always store the reference signal together with the input-output data

4. Using periodic excitations we can estimate simultaneously the FRF, the noise level, the level of the nonlinear distortions, and the level of the time-variation

5. Transient (time domain) \( \leftrightarrow \) leakage (frequency domain) = smooth function of the frequency

\[
Y(k) = G(j\omega_k)U(k) + T_G(j\omega_k)
\]

where \( T_G \) has the same poles as \( G \)

6. Never form the matrix product \( J^TJ \) when solving the normal equations; perform instead QR(\( J \)) or SVD(\( J \))

7. Using nonparametric noise models \( \Rightarrow \) errors-in-variables problems and identification in feedback are as simple as generalized output error problems

8. If the input is known and the output noise is an unobserved input, then the noise helps identifying the plant poles \( \Rightarrow \) use a parametric noise model (ARMAX model structure)
10. Books on system identification


## 11. Evaluation Matrix

<table>
<thead>
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<th>Insight</th>
<th>Application</th>
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<td>Chapter 2</td>
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<td>Chapter 10</td>
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12. Examples Questions

1. Explain and compare the fast and robust methods for detecting and quantifying the nonlinear distortions in frequency response function estimates.

2. Define the concept of the “best linear approximation” of a nonlinear dynamical system. For which class of excitation signals and which class of systems does it exist? What are the properties of the best linear approximation?

3. Explain the properties “consistent estimate” and “(asymptotically) unbiased estimate”. Which properties is most useful in system identification? Does asymptotic unbiasedness imply consistency? Give an example where (asymptotic) unbiasedness is the property needed.

4. What kind of model validation and model selection tools do you know? Discuss their pros and cons.

5. What is the basic issue of identification in feedback? How can it be solved? Discuss the different options. What if the system behaves nonlinearly? When is identification in feedback relevant?